For example, if one \( K' \in \mathbb{R}^{C'} \) and another \( K'' \in \mathbb{R}^{[C' : \text{the first column of } T']} \), and if \( A - BK \) is stable for both \( K' \)’s, then the corresponding compensator order can be zero and one, respectively.

Remark 6—Grand Design Algorithm: Highest Feedback System Performance and Robustness Achievable By State Feedback Control with Guaranteed Stability: When both high performance and robustness are desired, we must set up design priorities because performance and robustness are always contradictory to each other. We let stability (of \( A - BK \)) be the first and be the property which must be satisfied. We let robustness realizability [or (5)] and stronger control (which implies higher performance and robustness) of (11) (or higher \( rr \)) be the second and third design priorities, because the strength of this control would be very much lost if the critical robustness property of this control could not be actually realized.

Based on this design priority, we will let the control (11) be strong enough (or let \( rr \) be high enough) for the stabilization of \( A - BK \), but we will also let the value of \( rr \) be low enough when (5) is not satisfied. Now the following simple results can be obtained based on Remarks 2–5.

We let \( rr = r + m \) if (14) or (15) is guaranteed. This implies the strongest possible state feedback control (in achieving high performance and robustness) whose robustness properties are still fully realizable, with guaranteed feedback system stability.

If (15) is not satisfied and \( A - BK \) is unstable, then a higher \( rr = n + p + 1 \) or \( rr \geq (n + 1)/p \) (16) can guarantee (14) and hence the feedback system stability. In this case the free parameters \( e_i \) (\( i = r + 1, \ldots, rr - m \)) will be computed such that \( C \) is full row rank \([= rr \text{ of (16)}]\) while \([|e, D, B|]\) [or (5)] is minimized in the least squares sense.

The \( rr \) value of (16) may be lowered by using the freedom \( e_i \) (\( i = r + 1, \ldots, rr - m \)) for the stabilization of \( A - BK \) (instead of for the minimization of \([|e, D, B|]\)). The ascending order of \([|D, B|]\) [\( i = r + 1, \ldots, rr - m \)] (see the end of Step 4) is used to keep the corresponding \( rr - m \) for \( n \) nonzero rows of \( TB \) [or of (5)] at minimum in such a design.

IV. Conclusion

Remarks 3–5 (guaranteed feedback system stability, simple solution to strong stabilization, and low-order stabilizing output feedback compensator) are new claims. These claims are simply achieved based on (11) and on the significant results of Remarks 1–2. The first five remarks together enabled Remark 6—a significantly more general, simple, and tractable design (no numerical iteration in design computation) and the highest performance and robustness realizable by state feedback control with guaranteed stability.

References


A Transformed Luré Problem for Sliding Mode Control and Chattering Reduction

Steven Ching-Yei Chung and Chun-Liang Lin

Abstract—The sliding mode control problem is investigated here with the frequency domain approach. The authors show that the sliding mode control can be transformed into a Luré problem. With this formulation, it is shown that the conventional sliding mode control with a nonlinear sign function can only guarantee to make the overall system stable but not asymptotically stable. The authors also consider a practical situation where the sign nonlinearity may contain a hysteresis loop. For this situation, they show that even if the hysteresis loop is very small, there still exists a limit cycle. However, chattering phenomenon caused by the limit cycle behavior can be eliminated by a certain class of nonlinearities.
be eliminated by a wider class of nonlinearities, and this class of function for replacing the sign nonlinearity [9]–[11]. By applying very small. Using describing function and Typpskin’s method we are frequency, small-amplitude limit cycle even if the hysteresis loop is sign function contains a hysteresis loop, there will exist a high-
tem
domain. With this formulation, we transform the sliding mode control

III. Problem Formulation

The sliding mode control based on the state-space formulation is presented in this section. First let us consider the state-space representation

where $G(s) = C(sI-A)^{-1}B$. The equivalent control $u_{eq}$ is chosen such that

$$
\dot{h} = C\dot{x} = CAx + CBu_{eq} = 0. 
$$

Assume that $CB$ is invertible, we then obtain

$$
u_{eq} = -(CB)^{-1}C Ax.
$$

Let the control $u$ be divided into two parts as

$$
u = u_{eq} + V
$$

where $V$ is the additional control that makes the phase trajectory close to the sliding surfaces. Substituting (5) and (6) into (2), we have

$$
\dot{x} = (I - B(CB)^{-1} C) Ax + BV
$$

$$
\dot{h} = CX.
$$

The additional control $V$ is chosen in the following form:

$$
V(t) = (CB)^{-1}V_1(t)
$$

where

$$
V_1(t) = -K_0\phi(h, t)
$$

with the gain matrix $K \in \mathbb{R}^{m \times m}$ and $\phi(\cdot, \cdot)$ being some nonlinear function.

In the area of research for sliding mode control, $\phi(h, t)$ is often chosen to be a sign function. For this situation, $u$ becomes

$$
u = (CB)^{-1}C Ax - (CB)^{-1}K \cdot \text{sgn}(h).
$$

The relation between the sliding surfaces and the additional control can now be established

$$
H(s) = \tilde{G}(s) V(s)
$$

where $\tilde{G}(s) = C(sI - \tilde{A})^{-1}B$ with $\tilde{A} = (I - B(CB)^{-1} C) A$.

Remark: It is easy to show that the projection matrix $P = (CB)^{-1} C$ satisfies $(I - P) P = 0$.

The following theorem can be proved as in [3], [14], and [19].

Theorem 1: For the linear time-invariant system (2) with the sliding mode control (9), the following identity holds:

$$
det(sI - \tilde{A}) = s^m(CB)^{-1}, \quad det\begin{pmatrix} sI - A & B \\ C & 0 \end{pmatrix}.
$$

The result of Theorem 1 implies that the sliding surfaces should be chosen such that the zero dynamics (or transmission zeros) are stable.

Theorem 2: The following identity holds for the sliding mode control system (2) and (9):

$$
C(I - \tilde{A})^{-1} B (CB)^{-1} = \frac{1}{s} I_m.
$$

Proof: From the Leverrier’s algorithm [18], we have

$$
C(sI - \tilde{A})^{-1} B = \frac{1}{\Delta(s)} \left[ C Bs^{n-1} + CR_1 Bs^{n-2} + \cdots + CR_{n-1} B \right]
$$

where $\Delta(s) \equiv det(sI - \tilde{A}) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$ and

$$
R_1 = \tilde{A} + a_1 I
$$

$$
R_2 = AR_1 + a_2 I = \tilde{A}^2 + a_1 \tilde{A} + a_2 I
$$

$$
r_{n-1} = \tilde{A} R_{n-2} + a_{n-1} I = \tilde{A}^{n-1} + a_1 \tilde{A}^{n-2} + \cdots + a_{n-1} I.
$$

In addition, from the fact $\tilde{A} B = \tilde{C} (\tilde{A} - B(CB)^{-1} CA) B = (C - C)AB = 0$, we can have

$$
C \tilde{A} B = 0, \quad k > 0, \quad k = 1, 2, \cdots.
$$


This implies
\[ C R_1 B = C \hat{A} B + a_1 C B = a_1 C B. \]

Similarly, we have
\[
CR_k B = C \hat{A}^k B + a_1 C \hat{A}^{k-1} B + \cdots + a_k C B = a_k C B, \\
k = 1, 2, 3 \ldots.
\]

Thus
\[
C(sI - \hat{A})^{-1} B(C B)^{-1} = \frac{1}{\Delta(s)}(s^{n-1} + a_1 s^{n-2} + a_2 s^{n-3} + \cdots + a_{n-1}) I_m. \\
(15)
\]

On the other hand, from the well-known Cayley–Hamilton theorem, we have
\[
\hat{\hat{A}}^n + a_1 \hat{\hat{A}}^{n-1} + \cdots + a_{n-1} \hat{\hat{A}} + a_n I_m = 0.
\]

Premultiplying \(C\) and postmultiplying \(B\) to the above equation, we can obtain
\[
a_n C B = 0.
\]

Since \(C B\) is nonsingular, this implies \(a_n = 0\). We finally arrive at
\[
\Delta(s) = s(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}). \\
(16)
\]

Substituting (16) into (15) gives the desired result. This completes the proof.

IV. A TRANSFORMED LURÉ PROBLEM

The following result shows that with the formulation (9), (10), the sliding mode control system can be transformed into a Luré problem.

**Corollary 1:** The sliding mode control system can be transformed into a Luré problem of Fig. 1.

**Proof:** Using the representation (12), the block diagram of the overall system can be described by Fig. 1 where the linear transfer function is
\[
\tilde{G}(s) = K/s. \tag{17}
\]

The overall system is in a standard form of the closed-loop scheme discussed in the Luré problem.

For the sake of simplicity, the gain matrix \(K\) is often chosen in a diagonal form
\[
K = \text{diag}(k_1, k_2, \ldots, k_m). \quad k_i > 0, \quad i = 1, 2, \ldots, m. \tag{18}
\]

With the feedback gain matrix, the sliding mode control can be decoupled into several single-input/single-output (SISO) Luré problems as shown in Fig. 2 in which the linear transfer functions are \(k_i/s, i = 1, 2, \ldots, m\). If \(\phi(h, t)\) in (8b) is chosen to be a sign nonlinearity and \(K\) is given by (18), then
\[
V_1 = [-k_1 \cdot \text{sgn}(h_1) \quad -k_2 \cdot \text{sgn}(h_2) \quad \cdots \quad -k_m \cdot \text{sgn}(h_m)]^T. \tag{19}
\]

The sliding surface \(h = [h_1, h_2, \ldots, h_m]^T\) in Fig. 2 can be regarded as the output. Stability of the feedback system discussed in the Luré problem can usually be determined from passivity theory. Since the transfer function (17) is positive real, it follows from passivity theory that the closed-loop system is stable [10]. This means that the reaching condition can be attained [5], [10]. However, this cannot guarantee the globally asymptotic stability and exponential stability.

V. CHATTERING PHENOMENON

In practice, the sliding mode control with the switching nonlinearity always results in a chattering phenomenon. This is due to the fact that the switching action virtually involves a hysteresis in the nonideal hardware implementation (since in practice switching is not instantaneous and the value of the switching state variable \(h\) is not known with infinite precision [10]). In the following, we show that under this situation there will exist a high-frequency limit cycle.

A. Approximate Analysis by Describing Function Method

Let \(N(A, \omega)\) denote the describing function of the nonlinearity. For the sign function, the corresponding describing function can be obtained as [1]
\[
N(A) = 4/\pi A. \tag{20}
\]

The Nyquist diagram of the transformed SISO Luré problem is shown in Fig. 3. It can be seen that the loci of \(-1/N(A)\) and \(\tilde{G}(j\omega)\) intersect at \(\omega_c = \infty\) and \(A_c = 0\). This means that the system is stable. Let us consider a practical situation in which the sign function is not so perfect that it contains a hysteresis loop shown as in Fig. 4(a). The describing function for this nonlinearity is [1]
\[
N(A) = \frac{4}{\pi A} \sqrt{1 - \left(\frac{d}{A}\right)^2} - \frac{4d}{\pi A^2} j. \tag{21}
\]

where \(d\) is the size of the hysteresis loop. For this case, the Nyquist diagram is shown in Fig. 4(b). It can be observed that for a specific
Fig. 4. (a) A nonideal sign function with a hysteresis loop and (b) Nyquist diagram for (a).

there is an intersection at \( A_c = \frac{d}{\pi k_i} \). This implies that the limit cycle period is

\[
T = \frac{\pi^2 d}{2k_i}.
\]  

(22)

In general, \( d \) is very small; thus, this will be a high-frequency limit cycle.

**Remarks:**

1) The previous observation for the existence of a limit cycle is not restricted to memoryless nonlinearities.

2) The describing function analysis is usually regarded as an approximate method. This method is more applicable to the situation where the nonlinear block in Fig. 1 has the low-pass property. Therefore, the prediction of the limit cycle given above may not be sufficiently accurate. However, existence of a limit cycle can still be verified using the results derived in [6], [7], and [13]. That is, if there is an intersection at \( (A_c, \omega_c) \) in the Nyquist diagram and if the regularity condition \( \partial N(A)/\partial(A) \big|_{A_c} \neq 0, \partial d/\partial \omega \Im \tilde{G}(j \omega) \big|_{\omega_c} \neq 0 \) is satisfied, then there will exist a limit cycle with the first component of Fourier series being \( A \cdot \sin \omega t \), where \( \omega \in [\omega_c, \omega] \), \( A \subseteq [A_u, A_1] \), and \( \omega_u, \omega_1, A_u, A_1 \in R^+ \) are defined appropriately.

**B. Analysis by Typskin’s Method**

Since the describing function analysis is an approximate method, the period indicated in (22) is also an approximation. In this subsection, the Typskin’s method [1] is applied to obtain an exact limit cycle period.

The Typskin’s function for an antisymmetric relay-type device (which is periodically switching from one state to another state) is given by [1]

\[
\Lambda(\omega) = \frac{4}{\pi} \sum_{k=0}^{\infty} \left\{ \Re \tilde{G}_r(s) + j \frac{\Im \tilde{G}_r(s)}{2k + 1} \right\} e^{-j(2k+1)\omega}
\]

(23)

where \( \tilde{G}_r(s) = k_i/s \) is the scalar transfer function of the linear part in Fig. 2. For the sign nonlinearity with hysteresis, the Typskin’s condition is [1]

\[
\Re \Lambda(\omega) = -d, \quad \Re \Lambda(\omega) \leq \rho / \omega
\]

(24)

where \( \lim_{\omega \to \infty} s \tilde{G}_r(s) = \rho \) and \( d \) is the size of the hysteresis loop. There will exist a limit cycle for the feedback system described in Fig. 1, if the sign nonlinearity having hysteresis satisfies (24).

Now consider the SISO decoupled Luré problem shown in Fig. 2. By substituting \( \tilde{G}_r(s) = k_i/s \) into (23), the Typskin’s function of the transformed SISO Luré problem can be written as

\[
\Lambda(\omega) = 4k_i/j \pi \omega \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \right)
\]

(25)

It follows from \( \pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \cdots \), we have

\[
\Lambda(\omega) = -j \frac{k_i}{2\pi \omega}.
\]

(26)

It is easy to show that the Typskin’s condition is satisfied for

\[
\frac{k_i}{2\pi \omega} = d, \quad \Re \Lambda(\omega) \leq \rho / \omega
\]

(27)

where \( \rho = k_i \). Under this condition, there will exist a limit cycle with the period

\[
T = \frac{4d}{k_i}.
\]

(28)

Comparing (22) to (28), it can be found that the describing function method generates about 20% error.

**VI. CHATTERING-FREE DESIGN**

The saturation or sigmoid function is commonly adopted in the conventional sliding control to avoid the chattering phenomenon. This section shows that for a certain class of nonlinearities, the overall system is globally exponentially stable.

**Definition 1:** The nonlinearity \( \phi(h, t) : R^m \times R^+ \rightarrow R^m \) is said to belong to the sector \([a, b]\), \( a, b \in R \), if: 1) \( \phi(0, t) = 0 \) and 2) \( [\phi(h, t) - ah]^T[\phi(h, t) - bh] \leq 0 \), \( \forall t \geq 0, h \in R \). The nonlinearity is said to satisfy the finite sector condition, if \( \phi(h, t) \in [a, b] \) and \( 0 \leq a < b < \infty \).

**Theorem 3:** If the nonlinearity satisfies the finite sector condition and the gain matrix \( K \) is positive definite, then the system shown in Fig. 1 is globally exponentially stable.

**Proof:** The proof can be deduced from multi-input/multi-output (MIMO) circle criterion. It has been proved that with the sliding mode control (9), the overall system can be transformed into the structure shown as in Fig. 1. We now perform a similar transformation with respect to the positive definite matrix \( K \), i.e.,

\[
K = W^{-1} \tilde{A} W
\]

(29)

where \( W \) is a transformation matrix and

\[
\tilde{A} = \text{diag} (\lambda_1, \lambda_2, \cdots, \lambda_m)
\]

with \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \) being the eigenvalues of \( K \). The loop transfer matrix can then be written as

\[
\tilde{G}_s(s) = \tilde{G}(s) \cdot [I + a \tilde{G}(s)]^{-1}
\]

(30)

where \( a \) is the lower bound of the sector and \( \tilde{A} = \text{diag} (\lambda_1/s + a\lambda_1, \lambda_2/s + a\lambda_2, \cdots, \lambda_m/s + a\lambda_m) \). It is obvious that \( \tilde{G}_s(s) \) is a stable transfer function matrix. Furthermore, we have

\[
\inf_{s \in R} \lambda_{\text{min}} \left\{ \frac{1}{2} \left[ \tilde{G}_s(j\omega) + \tilde{G}_s(-j\omega) \right] \right\} + \frac{1}{b-a}
\]

\[
\geq \inf_{s \in R} \lambda_{\text{min}} \left\{ \Re \tilde{G}_s(j\omega) \right\} + \frac{1}{b-a}
\]

(31)

\[
= \inf_{s \in R} \left\{ \frac{\lambda^2}{a^2 \lambda^2 + \omega^2} \right\} + \frac{1}{b-a} = \frac{1}{b-a} > 0
\]

According to MIMO circle criterion [13], we can conclude that the system in Fig. 1 is globally exponentially stable.

With the control law (9), if the nonlinearity satisfies the finite sector condition and the gain matrix \( K \) is positive definite, then the sliding mode control can attain the reaching condition. This means that the phase trajectory will converge to the sliding surface.
Remarks:
1) Note that elements of the closed-loop output shown in Fig. 1 are the variables of sliding surfaces, not the system outputs.
2) The saturation and sigmoid functions belong to the class of nonlinearities defined above.
3) For SISO systems, inequality (31) implies that the Nyquist plot of $\tilde{G}_{\alpha} (j\omega)$ does not enter the disk $D(a, b) = \{ z \in C \mid | z + 0.5 (1/a + 1/b) | \leq 0.5 | 1/a - 1/b | \}$.
4) Theorem 3 indicates that the gain matrix $K$ need not be diagonal.

For further ensuring the global exponentially stability, we give the following theorem.

**Theorem 4:** Consider the linear time-invariant system (2) with the sliding mode control (9). The overall closed-loop system is globally exponentially stable, if:
1) the sliding surface is chosen such that the transmission zeros are stable;
2) the nonlinearity $\varphi (h, t)$ satisfies the finite sector condition;
3) the nonlinearity $\varphi (h, t)$ is Lipschitz continuous in $h$ and uniformly for all $t \geq 0$.

**Proof:** If the control law is described by (9), from Corollary 1 we have
$$\dot{h} = -K \cdot \varphi (h, t).$$ (32)

Using the coordinate transformation and by the definition of zero dynamics [10], [13], the realization of (12) can be expressed as
$$\ddot{z} = Q \ddot{z} + P \cdot h$$ (33)
where $P, Q \in R^{(n-m) \times (n-m)}$, the eigenvalues of $Q$ are the transmission zeros of the transfer function matrix $G(s)$. When $h = 0$, (33) becomes
$$\ddot{z} = Q \ddot{z}$$ (34)
which corresponds to the zero dynamics of the system (33).

It can be observed that (32) and (33) are in the form of (1), and (32) and (34) correspond to the isolated subsystems. Since $||P|| \leq M ||Q|| \leq M$, and the function $\varphi (h, t)$ is Lipschitz continuous in $h$ and uniformly for $t > 0$, Assumptions 2) and 3) are satisfied. From Theorem 3, we know that the isolated subsystem (32) is globally exponentially stable. In addition, under Assumption 1), the transmission zeros of the transfer function matrix $G(s)$ are stable. Therefore, the isolated linear subsystem (34) is also globally exponentially stable. From Lemma 1, we can conclude that the overall system (32)–(34) is globally exponentially stable.

**Remark:** If the nonlinearity $\varphi (\cdot)$ is time-invariant, the assumption for $\varphi (\cdot)$ being Lipschitz continuous is not necessary.

Suppose that the sign function is replaced by a saturation or sigmoid function to reduce the chattering phenomenon, i.e., as shown in (35a) or (35b), shown at the bottom of the page, where $\text{sigmoid} (h/\varepsilon) = \tanh (h/\varepsilon)$. It can be observed that the sector for the saturation function is $[0, 1/\varepsilon]$. Also, it is easy to show that slope of the sigmoid function is constrained by
$$0 \leq \frac{d}{dh} \tanh \left( \frac{h}{\varepsilon} \right) \leq \frac{1}{2 \varepsilon}.$$

These functions are time-invariant and satisfy the finite sector condition. According to Theorem 4, the sliding mode control system with these nonlinearities is globally exponentially stable.

We give the following corollary as a conclusion.

**Corollary 3:** If the sliding surface is chosen such that the transmission zeros are stable, the overall system (4) and (9) will be globally exponentially stable.

**VII. AN ILLUSTRATIVE EXAMPLE**

Consider a linear time-invariant system with the following system matrices:
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3.199 & -3.58 & -3.3 & 0 \\ 0.001 & 0.02 & 0.1 & 1 \end{bmatrix}^T,$$
$$B = [0 \ 0 \ 0 \ 0]^T.$$
$$C = [1 \ 0 \ 0 \ 0].$$

The corresponding transfer function can be expressed as
$$G(s) = \frac{0.001 s^3 + 0.02 s^2 + 0.1 s + 1}{s^4 + 3.2 s^3 + 3.6 s^2 + 3.3 s}.$$

Related output responses for the closed-loop scheme are shown in Figs. 5–9. In Fig. 5, the control (9) with the sign nonlinearity results in chattering phenomenon. The chattering phenomenon can be eliminated by using a class of nonlinearities that satisfies the finite sector condition. As shown in Fig. 6, the chattering phenomenon has been removed using a saturation function with the boundary layer $\varepsilon = 0.01$. In addition to the saturation function, the chattering phenomenon can also be eliminated by a sigmoid nonlinearity or a special nonlinearity shown in Fig. 8. The latter is nonsymmetric, but it satisfies the finite sector condition with the sector $[0, 100]$. Simulation results for these cases are shown in Figs. 7 and 9.
VIII. CONCLUSION

The conventional sliding mode control problem has been transformed into a frequency-domain Luré problem. We show that if the sign nonlinearity is not perfect so that there exists a hysteresis loop, then no matter how small the hysteresis loop is, there will be a limit cycle. For the sliding mode control, this limit cycle behavior induces a chattering phenomenon. The limit cycle period has been explored by using the describing function method and Typskin’s method. By the circle criterion, we also show that if the nonlinearity satisfies the finite sector condition and it is Lipschitz continuous, then the overall system will be globally exponentially stable.

ACKNOWLEDGMENT

The authors deeply appreciate the anonymous reviewers for their valuable comments.

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