Lyapunov Redesign of Analog Phase-Lock Loops

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Abstract—In general, the design of phase-lock loops has been done by a combination of linear analysis, phase plane plots, rule of thumb, and simulation. Very few analytical tools have been used to determine the stability of the nonlinear models used for these devices. A method from the control literature known as Lyapunov redesign [1] has recently been used to design a third-order phase-lock loop whose nonlinear model is guaranteed to be stable [2]. In this paper, this technique is demonstrated to be an effective stability analysis and design technique for many analog phase-lock loops. The ability of loops designed using these techniques to track a phase step is also proven.

I. INTRODUCTION

ANALOG phase-lock loops have been around for many years as noted in [3] and [4]. While the field is considered quite mature and there have been many books written on the subject, e.g. [5]–[7], very little has been said in the commonly available literature about the stability analysis of the nonlinear model of the analog phase-lock loop. There have been a few examples of authors using Lyapunov analysis on PLL’s [8]–[10], but in general, the analysis and design of phase-lock loops has been done by a combination of linear analysis, phase plane plots, rule of thumb, and simulation [3]–[7]. A method from the control literature known as Lyapunov redesign [1] has recently been used to design a third-order phase-lock loop whose nonlinear model is guaranteed to be stable [2]. In this paper, this technique is demonstrated to be an effective stability analysis and design technique for many analog phase-lock loops.

Typically, stability analysis deals with noise free models. This will be the case here. While practical analysis and simulation of a real system must include some noise model, a prerequisite for such analysis is either the knowledge or the assumption that the loop is stable. This paper will attempt to provide the former. Furthermore, stability analysis deals with the homogeneous (no input) differential equation of the system. This will be useful here, as it is often convenient to perform block diagram manipulations to give a set of states for which it is easy to extract stability results. Finally, stability is an asymptotic property. Knowing that a system is stable or that it tracks a step input does not in itself yield performance results. The purpose of this paper is to provide a design method for analog PLL’s that will guarantee stability and tracking.

The structure of this paper is as follows. Section II will review the model of an analog PLL. Section III will introduce the necessary definitions and theorems for doing analysis. The actual analysis as applied to phase-lock loops will be done in Section IV.

II. THE MODEL OF AN ANALOG PLL

The block diagram of an analog phase-lock loop (PLL) is shown in Fig. 1. While this block diagram is close to the actual implementation of the PLL, several steps are normally taken to facilitate analysis. Using the familiar trigonometric identity in terms of the

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1Digital phase-lock loops are denoted by DPLL.

While this is useful for studying loops that are near lock, it does not help for analyzing the loop when \( \psi \) is large.

2) Phase plane portraits. The disadvantage to this is that phase plane portraits can only completely describe first and second order systems.

3) Simulation.

In this paper, the second method of Lyapunov will be proposed as a method for analyzing the stability and tracking of the nonlinear phase lock loop model. It will be shown that a fairly regular procedure known as Lyapunov redesign [1], which is quite well known in the area nonlinear control theory is quite applicable to solving this class of problems.

III. LYAPUNOV STABILITY

The second method of Lyapunov [11] is commonly used in stability analysis of nonlinear differential equations because it does
not require the solution to the differential equation. A very intuitive discussion of this can be found in Ogata [12]. The second method of Lyapunov is based on the generalized energy in the system. If an energy like function of the system state (i.e., a positive definite function of the state which is nonvanishing as long as the state is nonzero) is found which is constantly decreasing, then the system is asymptotically stable. A general form of a vector differential equation is

\[ \dot{x} = f(x, t) \text{ where } x, \dot{x} \in \mathbb{R}^n. \]  

(2)

An equilibrium state is any state such that

\[ f(x_e, t) = 0. \]  

(3)

Usually, a transformation is made so that the origin of state space is an equilibrium state, i.e.,

\[ f(0, t) = 0. \]  

(4)

Theorem I (LaSalle’s Theorem): [11] For the system defined by (2), suppose there exists a positive definite scalar function of \( x \), \( V(x) \), such that \( V(x) \) is negative semidefinite, i.e.,

\[ V(x) > 0, \quad \dot{V}(x) \leq 0 \quad \forall x \neq 0 \]  

(5)

Suppose also that the only solution of \( \dot{x} = f(x, t) \), \( \dot{V}(x) = 0 \) is \( x(t) = 0 \) for all \( t \geq 0 \). Then \( \dot{x} = f(x, t) \) is globally asymptotically stable.

This theorem will prove to be quite useful in the next section. In general, \( V \) is known as a Lyapunov function if it satisfies either LaSalle’s Theorem or Lyapunov’s Main Stability Theorem [11]. It turns out in practice that Theorem I is often easier to satisfy. Another definition that is necessary is that of a sector nonlinearity.

Definition I (Sector Nonlinearity): The function \( f(\cdot, \cdot) \) is said to belong to sector \( [\alpha, \beta] \) if

\[ a\psi^2 \leq \psi f(\psi, \gamma) \leq \beta\psi^2 \quad \forall \psi \in \mathbb{R}, \quad \forall \gamma \geq 0. \]

In other words, a sector nonlinearity would belong to sector \( [\alpha, \beta] \) if it fell in the dotted region of Fig. 3. Lyapunov redesign starts with a candidate Lyapunov function. The function is parameterized by the design parameters of the system in question. These parameters are then chosen so that the candidate Lyapunov function meets the requirements of either Lyapunov’s Theorem or LaSalle’s Theorem. In this paper, that function will have the form

\[ V = \int_{0}^{x} f(\sigma) d\sigma + x^TPx \]  

(6)

which was introduced by LaSalle and Lefschetz [13]. \( P \) is a positive definite matrix, \( x \) is some portion of the system state, and \( f(\cdot) \) is a nonlinearity which lies in sector \([0, \infty]\). That is to say

\[ 0 \leq f(\sigma) \sigma. \]  

(7)

The key is to satisfy conditions such that \( V \geq 0 \), but

\[ \dot{V} = f(z) \dot{z} + x^TPx \leq 0. \]  

(8)

In the case of the analog PLL, \( f(\sigma) = \sin(\sigma) \) and it is fairly easy to see that this is in sector \([0, \infty]\) for \(-\pi < \sigma < \pi \). Also, in the cases when \( P \) is a \( 2 \times 2 \) matrix (third-order PLL’s) the conditions for \( P > 0 \) are

\[ p_{11} > 0, \quad p_{11}p_{22} > p_{12}^2, \quad \Rightarrow p_{22} > 0. \]  

(9)

IV. NONLINEAR ANALYSIS OF PLL’S

In this section, the above Lyapunov analysis techniques are applied to a variety of phase-lock loops. The structure of this section will be a series of examples of second- and third-order phase-lock loops, some with zeros and some without. In each case, the differential equations and a corresponding block diagram will be shown. A Lyapunov function is chosen which is parameterized by the parameters of the analog phase-lock loop. Conditions for stability of the PLL are then determined in terms of the loop parameters. Finally, using these same loop parameters, the tracking of step inputs to the PLL is proven.

A. Second-Order PLL with No Zeros

A second order PLL with no zeros is shown in Fig. 4. The differential equations corresponding to this loop with no external inputs \((\theta = \gamma = 0)\) are

\[ \dot{x} = K_p \sin \psi - a_1 x \text{ and} \]  

(10)

\[ \dot{\psi} = -K_1(x + \gamma) = -K_1x \]  

(11)

where \( \psi = \theta - \phi = -\phi \). Choose

\[ V = \int_{0}^{\psi} \sin(\sigma) d\sigma + \frac{1}{2} p\dot{x}^2, \quad p > 0. \]  

(12)

The term under the integral is positive for \(-\pi < \psi < \pi \) and this fact will be used quite often. Then

\[ V = \sin(\psi) \dot{\psi} + p\dot{x}^2 \]  

(13)

\[ = px(K_p \sin \psi - a_1 x) + \sin(\psi)(-K_1 x) \]  

(14)

\[ = -a_1 p\dot{x}^2 + \sin(\psi) x(pK_p - K_1). \]  

(15)

In order to invoke LaSalle’s Theorem, we must have \( V'(\psi, x) \geq 0 \) with \( V = 0 \Leftrightarrow \psi = x = 0 \) and \( V \leq 0 \). Assuming \( \psi \in (-\pi, \pi) \), the condition for \( V \geq 0 \) is

\[ p > 0. \]  

(16)

The conditions that guarantee \( V \leq 0 \) are

\[ a_1 p > 0, \]  

(17)

\[ pK_p - K_1 = 0. \]  

(18)

For \( K_p, K_1 > 0 \) it is always possible to satisfy conditions (18) and (16) by picking

\[ p = \frac{K_p}{K_1}. \]  

(19)
Condition (17) is easy to satisfy by picking
\[ a_1 > 0, \]  
which leaves
\[ V = \int_0^\psi \sin(\sigma) \, d\sigma + \frac{1}{2} K_v x^2 \quad \text{and} \]
\[ \dot{V} = -a_1 \frac{K_v}{K_p} x^2 \leq 0. \]  
Finally, the only values for \( \psi \) and \( x \) which results in \( V = \dot{V} = 0 \) is \( \psi = x = 0 \).

1) Tracking a Phase Step: The second-order PLL with no zeros designed above is stable. It will now be shown that this loop can also track a step input. The equations for the PLL corresponding to Fig. 4 with an input at \( \theta \) are
\[ \dot{x} = K_p \sin \theta - a_1 x, \]  
\[ \psi = \theta - \phi, \]  
\[ \dot{\psi} = \theta - K_v(x + \gamma) = \theta - K_v x. \]  
As above, choose the Lyapunov function
\[ V = \int_0^\psi \sin(\sigma) \, d\sigma + \frac{1}{2} K_v x^2 \]  
where the choice of \( p \) from (19) has as been made and \( a_1 > 0 \). Then
\[ \dot{V} = \sin(\psi) \dot{\theta} - a_1 \frac{K_v}{K_p} x^2 \]  
where the second term is the same as the no input case, (22), and the first term corresponds to excitation caused by the input to \( \theta \). Now say \( \theta \) is a step. Then
\[ \theta = \theta_0 \delta(t) \]  
where \( \delta(t) \) is the impulse function. Integrating (27) forward in time and noting that \( \sin \psi(0^+) = \sin \theta_0 \) yields (due to the sifting property of the impulse function):
\[ V(t) = \int_0^t \sin(\psi) \, d\theta - a_1 \frac{K_v}{K_p} \int_0^t x^2 \, dt \]  
\[ = \theta_0 \sin \theta_0 - a_1 \frac{K_v}{K_p} \left[ t \right]_0^t x^2 \, dt. \]  
The first term of (30) is a positive constant for \( -\pi < \theta_0 < \pi \). The second term is a negative number which will grow without bound unless \( x \) goes to 0. If \( x \) did not converge to 0, then \( V(t) \) would eventually become negative which is impossible since \( V(t) \) was chosen to be a positive definite function. Thus, \( x \) must converge to 0. As in the discussion of stability, the only value of \( \psi \) for which \( x \) can remain identically 0 is \( \psi = 0 \), thus the loop must track a phase step input.

B. Second-Order PLL with One Zero
A second-order PLL with one zero is shown in Fig. 5. In this case, the analysis is simplified a bit if the loop is redrawn as in Fig. 6. These two loops are equivalent from the point of view of closed-loop stability. The second configuration merely makes it possible to draw some new state variables out of the block from \( x \) to \( \phi \) as shown in Fig. 7. The differential equations corresponding to this loop with no external inputs (\( \theta = \gamma = 0 \)) are:
\[ \dot{\psi} = K_v b_1 x \]  
\[ \phi = K_v x + y \]  
\[ \dot{x} = K_p \sin \phi - a_1 x \]  
\[ \dot{\phi} = K_v b_1 x - K_v a_1 x \]  
where \( \psi = \theta - \phi = -\phi \). Choose
\[ V = \int_0^\psi \sin(\sigma) \, d\sigma + \frac{1}{2} p x^2, \quad p > 0. \]  
Using analysis entirely analogous to that in Section IV-A and choosing \( K_p, K_v > 0 \) the conditions for satisfying LaSalle’s Theorem are
\[ p = K_v(b_1 - a_1) \quad \text{and} \]
\[ b_1 > a_1 > 0. \]  
This leaves
\[ V = \int_0^\psi \sin(\sigma) \, d\sigma + \frac{1}{2} K_v(b_1 - a_1) x^2 \]  
\[ \text{and} \]
\[ V = \int_0^\psi \sin(\sigma) \, d\sigma + \frac{1}{2} K_v K_p a_1 x^2 \]  
\[ \text{and} \]
Finally, the only values for $p$ and $x$ which results in $V = \dot{x} = \dot{\psi} = 0$ is $x = 0$.

It is interesting to note that if we chose $b_1 = a_1$ then the pole and zero from $x$ to $\phi$ would cancel leaving $x = \phi$. That is to say, the model can be reduced to that of a first-order analog phase-lock loop. For this new model, this same Lyapunov function works, except that $p = 0$ and thus

$$V = \int_0^t \sin(\sigma) d\sigma \quad \text{and} \quad \dot{V} = -K_s K_p [\sin \psi]^2 \leq 0. \quad (42)$$

1) Tracking a Phase Step: It turns out that a completely analogous result for tracking holds for the second-order PLL with one zero as shown for the second-order PLL with no zeros. As above, choose $V$ from (36) subject to (37) and (38) to yield

$$V = \sin(\psi) \dot{\theta} - K_s K_p [\sin \psi]^2 - a_1 K_s (b_1 - a_1) x^2 \quad (43)$$

where the last two terms are the same as the no input case, (40), and the first term corresponds to excitation caused by the input to $\theta$. As before $\dot{\theta}$ is a step, so integrating (43) forward in time yields [see (29)–(30) for details]

$$V(t) = \theta_0 \sin \theta_0 - K_s K_p \int_0^t [\sin \psi]^2 dt - a_1 \frac{K_s (b_1 - a_1)}{K_p} \int_0^t x^2 dt. \quad (44)$$

The first term of (44) is a positive constant for $-\pi < \theta_0 < \pi$. The latter terms are negative numbers which will grow without bound unless $\psi$ and $x$ must go to zero to preserve the positive definiteness of $V$ and thus this loop tracks a step input to $\theta$.

C. Third-Order PLL with No Zeros

A third order phase-lock loop with no zeros turns out to be unstable, even in the linearized (small signal) model. It turns out that the feedback structure can be stabilized by adding what is known in the controls literature as velocity feedback as shown in with the dashed line in Fig. 8. While this is useful from a theoretical point of view and in fact a Lyapunov function can be found for this system, this model no longer corresponds to a phase-lock loop and will not be pursued here.

D. Third-Order PLL with One Zero

A third-order analog PLL with one zero is shown in Fig. 9. Again, for the sake of stability and tracking analysis it is convenient to redraw the loop as in Fig. 10. From here the necessary state variables can be drawn out as in Fig. 11. The state equations corresponding to Figs. 10 and 11 are

$$\dot{x} = K_p \sin \psi - a_1 z, \quad (45)$$

$$\dot{y} = K_b_1 x, \quad \text{and} \quad (46)$$

$$\dot{\psi} = -\dot{\phi} = -K_s x - y. \quad (47)$$

Choose

$$V = \int_0^t \sin(\sigma) d\sigma + \frac{1}{2} [x^T P \dot{x}] \quad (48)$$

where $P$ is a symmetric, positive definite, $2 \times 2$ matrix. In order to invoke LaSalle’s Theorem, we must have $V(\psi, x, y) \geq 0$ with $V(0, \psi = x = y = 0 \text{ and } V \leq 0$. Assuming $\psi \in (-\pi, \pi)$ we can satisfy (9) with

$$P = \begin{bmatrix}
K_p & 1 \\
K_p & K_p \\
1 & a_1 \\
K_p & K_p K_s b_1 \\
\end{bmatrix}, \quad (49)$$

$$a_1 > b_1 > 0, \quad \text{and} \quad (50)$$

$$K_p, K_s > 0. \quad (51)$$

Thus, $V \geq 0$ and

$$\dot{V} = -x^T \left[ \frac{K_s}{K_p} (a_1 - b_1) \right] \leq 0. \quad (52)$$

Finally, the only place that $V$ and (53)–(55) can vanish is for $z = y = \phi = \dot{\psi} = 0$, so using LaSalle’s Theorem proves stability.

E. Third-Order PLL with Two Zeros

The block diagram of the PLL that was analyzed in full [2] is shown in Fig. 12. The results will be summarized here. This is a third-order PLL with two zeros. Note that an extra gain $K$ is involved in this model, but for consistency with the previous examples this could be set to 1 without loss of generality. For the sake of stability and tracking analysis it is convenient to redraw the loop as

In a mechanical analog of the analog PLL, $\phi$ would correspond to the position and $x$ would correspond to the velocity.
in Fig. 13. From here the necessary state variables can be drawn out as in Fig. 14. The state equations corresponding to Figs. 13 and 14 are

\[ \dot{z} = K_p \sin \psi - a_1 z, \quad (53) \]

\[ \dot{y} = K b_0 z, \quad (54) \]

\[ \dot{\psi} = -K_s K K_p \sin \psi - K_s K (b_1 - a_1) z - K_s y. \quad (55) \]

Choose

\[ V = \int_0^\psi \sin(\sigma) d\sigma + \frac{1}{2} \left[ z \ y \right]^T P \left[ \begin{array}{c} z \\ y \end{array} \right] \]

\[ (56) \]

where \( P \) is a symmetric, positive definite, \( 2 \times 2 \) matrix. In order to invoke LaSalle’s Theorem, we must satisfy \( V(\psi, y, z) \geq 0 \) with \( V = 0 \Rightarrow \psi = y = z = 0 \) and \( V \leq 0 \). Assuming \( \psi \in (-\pi, \pi) \) we satisfy (9) and guarantee \( V \geq 0 \) and \( V \leq 0 \) with

\[ P = \begin{bmatrix} K_s K & K_s \\ K_p & K_p \end{bmatrix} \]

\[ K_s K K_p > 0 \Leftrightarrow \frac{K_s K}{K_p} > 0, \quad K_p \neq 0, \quad (58) \]

\[ b_1 > a_1, \quad (b_1, a_1 \text{ same sign}) \] and

\[ b_0 - (b_1 - a_1) a_1 < 0, \quad (59) \]

\[ (60) \]

Finally, the only place that \( \dot{V} \) and (45)–(47) can vanish is for \( x = y = \phi = \psi = 0 \), so using LaSalle’s Theorem proves stability.

**F. Tracking for Third-Order Loops**

The tracking analysis of a step input for the third-order loops is completely analogous to that of the second-order loops and will be omitted here for brevity.

**V. DISCUSSION**

It is interesting to note that the conditions for stability of the nonlinear second-order models, i.e., \( K_s K_p > 0, \quad a_1 > 0 \), and (in the case of a zero) \( b_1 > a_1 \), correspond to the linear design rules. First \( K_s K_p > 0 \) guarantees negative feedback for the loop. The conditions on \( a_1 \) and \( b_1 \) (if present) correspond to the filter being a stable, low-pass filter. As the presence of a stable, low-pass filter is always assumed in the design of a PLL, this implies that all second-order PLL’s that are designed using the standard assumption in Section II are stable. Thus, the theory here reinforces what has been empirically observed by many designers over the years.

**VI. CONCLUSION**

This paper has endeavored to approach the problem of stability and tracking analysis for the nonlinear model of analog phase-lock loops in a novel manner. The theory used, Lyapunov analysis, and the particular method for design, Lyapunov redesign, are not new to control theory. Only their application to analog phase-lock loops are new. This method appears to be quite useful for this problem and its applicability seems to be quite broad.

**REFERENCES**


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