Bifurcation analysis in delayed feedback Jerk systems and application of chaotic control

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Abstract

Jerk systems with delayed feedback are considered. Firstly, by employing the polynomial theorem to analyze the distribution of the roots to the associated characteristic equation, the conditions of ensuring the existence of Hopf bifurcation are given. Secondly, the stability and direction of the Hopf bifurcation are determined by applying the normal form method and center manifold theorem. Finally, the application to chaotic control is investigated, and some numerical simulations are carried out to illustrate the obtained results.

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1. Introduction

A Jerk equation is an autonomous third-order differential equation of the form

\[
\frac{d^3 x}{dt^3} = f \left( \frac{d^2 x}{dt^2}, \frac{dx}{dt}, x \right).
\]  

(1.1)

Here, \( x \) denotes a scalar, for example the position coordinate. It is well known that any explicit \( n \)-order scalar ordinary differential equation can be recast in the form of a system of \( n \) coupled first-order differential equations, i.e. in the form of a \( n \)-dimensional dynamical system. In particular, a scalar third-order differential equation of the form (1.1) can be transformed into the three-dimensional dynamical system:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= x_3(t), \\
\dot{x}_3(t) &= f(x_1(t), x_2(t), x_3(t))
\end{align*}
\]

(1.2)

for letting

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\[ x_1(t) = x, \quad x_2(t) = \frac{dx_1}{dt}, \quad x_3(t) = \frac{dx_2}{dt}. \]

Since the creative works of [1], this model has recently attracted considerable interest [2–6] because it constitute an important tool to identify and classify elementary chaotic flows. Jerk equations constitute an interesting subclass of dynamical systems that can exhibit many major features of regular or chaotic behavior.

Time-delayed feedback has been introduced as a powerful tool for control of unstable periodic orbits or control of unstable steady states [7]. In the present paper, regarding the delay as parameter, we investigate the effect of delay on the dynamics of Jerk system with delayed feedback:

\[
\begin{align*}
\dot{x_1}(t) &= x_2(t), \\
\dot{x_2}(t) &= x_3(t), \\
\dot{x_3}(t) &= f(x_3(t), x_2(t), x_1(t)) + k(x_1(t) - x_1(t - \tau)),
\end{align*}
\]

where \( \tau \) is a positive constant number.

We first consider the effect of delay on the steady states, and then investigate the existence of local Hopf bifurcations. By using the normal form theory and center manifold argument, we derive the explicit formulas to determine the stability, direction and other properties of bifurcating periodic solutions. Finally, we give several numerical simulations, which indicate that when the delay passes through certain critical values, chaotic oscillation is converted into a stable steady state or a stable periodic orbit.

The paper is organized as follows. In Section 2, we analyze the distribution of the eigenvalues and obtain the stability and existence of Hopf bifurcations. In Section 3, the algorithm for determining the properties of Hopf bifurcations is obtained by using the normal form method and center manifold theorem. In Section 4, as an application, we give a delay feedback control example which is a chaotic Jerk system.

2. Stability analysis

In this section we shall analyze the distribution of the roots of the associated characteristic equation to discuss the stability and existence of Hopf bifurcation, by regarding \( \tau \) as parameter.

For (1.3), we make the following assumption:

\( (H_1) \ f \in C^2 \) and \( f(0,0,x^*) = 0 \).

From the assumption we know that \((0,0,x^*)\) is an equilibrium of Eq. (1.3). Let

\[
\begin{align*}
y_1(t) &= x_1(t) - x^*, \\
y_2(t) &= x_2(t), \\
y_3(t) &= x_3(t).
\end{align*}
\]

The Eq. (1.3) can be rewritten as

\[
\begin{align*}
\dot{y_1}(t) &= y_2(t), \\
\dot{y_2}(t) &= y_3(t), \\
\dot{y_3}(t) &= f(y_3(t), y_2(t), y_1(t)) + k(y_1(t) - y_1(t - \tau)).
\end{align*}
\]

The linearization of Eq. (2.1) around \((0,0,0)\) is

\[
\begin{align*}
\dot{y_1}(t) &= y_2(t), \\
\dot{y_2}(t) &= y_3(t), \\
\dot{y_3}(t) &= -a_3y_3(t) - a_2y_2(t) - (a_1 - k)y_1(t) - ky_1(t - \tau).
\end{align*}
\]

The associated characteristic equation is

\[
\det \begin{pmatrix}
\lambda & -1 & 0 \\
0 & \lambda & -1 \\
a_1 - k + ke^{-\lambda} & a_2 & \lambda + a_3
\end{pmatrix} = 0,
\]

that is

\[
\lambda^3 + a_3\lambda^2 + a_2\lambda + a_1 - k + ke^{-\lambda} = 0.
\]

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Consider the function
\[ g(z) = z^3 + (a_1^2 - 2a_2)z^2 + (a_2^2 - 2a_3(a_1 - k))z + a_1^2 - 2a_1k \]  
(2.4)
and
\[ g'(z) = 3z^2 + 2(a_1^2 - 2a_2)z + (a_2^2 - 2a_3(a_1 - k)). \]

If \( A = (a_1^2 - 2a_2)^2 - 3(a_2^2 - 2a_3(a_1 - k)) > 0 \), then \( g'(z) \) has two zeros \( z_1^1 = \frac{-a_1^2 - 2a_2 - \sqrt{A}}{3}, z_2^1 = \frac{a_1^2 - 2a_2 + \sqrt{A}}{3} \), and \( g''(z_1) < 0, g''(z_2) > 0 \). It follows that \( z_1^1 \) is the local maximum of \( g(z) \) and \( z_2^1 \) is the local minimum of \( g(z) \).

We make the assumption \((H_2)\) on \( g(z) \):\n\( (H_2) \) If \( g(z_0) = 0 \), then \( g'(z_0) \neq 0. \)

Following the discussion above, the preliminary results are obtained:

**Lemma 2.1.** Assume that \((H_2)\) holds.

1. Let \( a_1^2 < 2a_1k \).
   a. If \( g'(z) \geq 0 \) for all \( z > 0 \), then Eq. \((2.4)\) has only one positive zero \( z_0^{(1)} \), and \( g'(z_0^{(1)}) > 0 \).
   b. If \( z_1^1, z_2^1 \) are positive zeros of \( g'(z) \), and \( g(z_1^1) > 0, g(z_2^1) < 0 \), then Eq. \((2.4)\) has three positive zeros \( z_1^{(1)} \in (0, z_1^1), z_1^{(2)} \in (z_1^1, z_2^1), z_1^{(3)} \in (z_2^1, +\infty) \). In this case, \( g'(z_1^{(1)}) > 0, g'(z_1^{(2)}) < 0, g'(z_1^{(3)}) > 0 \).
2. Let \( a_1^2 > 2a_1k \).
   a. If \( g'(z) > 0 \) for all \( z > 0 \), then there is no positive zero of Eq. \((2.4)\).
   b. If \( z_1^1, z_2^1 \) are positive zeros of \( g'(z) \), and \( g(z_1^1) > 0, g(z_2^1) < 0 \), then Eq. \((2.4)\) has two positive zeros \( z_1^{(1)} \in (z_1^1, z_2^1), z_2^{(1)} \in (z_2^1, +\infty) \). In this case, \( g'(z_1^{(1)}) < 0, g'(z_2^{(1)}) > 0 \).

Let us consider the purely imaginary roots \( \pm i\omega \) of the Eq. \((2.3)\). If \( i\omega(\omega > 0) \) is a root of Eq. \((2.3)\), then
\[-i\omega^3 - \omega^2a_3 + i \omega a_2 + a_1 - k + ke^{-i\omega t} = 0.\]

Separating the real and imaginary parts, we have
\[
\begin{align*}
-\omega^3 + a_3\omega &= k \sin \omega t, \\
-a_2\omega^2 + a_1 - k &= -k \cos \omega t.
\end{align*}
\]
(2.5)

Hence,
\[
\omega^6 + (a_3^2 - 2a_2)\omega^4 + (a_2^2 - 2a_3(a_1 - k))\omega^2 + a_1^2 - 2a_1k = 0.
\]
(2.6)

Let \( z_i^{(j)} (i = 0, 1, 2, j = 1, 2, 3) \) be positive zeros of the equation \( g(z) = 0 \). Then \( \omega_i^{(j)} (i = 0, 1, 2, j = 1, 2, 3) \) are roots of Eq. \((2.6)\). Define \( (\omega_i^{(j)})^{(n)} \) as \((2.7)\) or \((2.8)\):
\[
(\omega_i^{(j)})^{(n)} = \frac{1}{\omega_i^{(j)}} \left( \arccos \left( \frac{a_3\omega_i^{(j)^2} - a_1 + k}{k} + 2n\pi \right) \right) \quad n = 0, 1, \ldots,
\]
(2.7)
or
\[
(\omega_i^{(j)})^{(n)} = \frac{1}{\omega_i^{(j)}} \left( \arcsin \left( -\frac{(a_3\omega_i^{(j)})^2 + a_2\omega_i^{(j)}}{k} + 2n\pi \right) \right) \quad n = 0, 1, \ldots
\]
(2.8)

Now let us consider the behavior of the roots of Eq. \((2.3)\) near the values \( \tau = (\omega_i^{(j)})^{(n)} (i = 0, 1, 2; j = 1, 2, 3; n = 0, 1, \ldots) \). To do this we assume that \( \lambda(\tau) = \alpha(\tau) + i\omega(\tau) \) is a solution of Eq. \((2.3)\) satisfying \( \alpha((\omega_i^{(j)})^{(n)}) = 0 \) and \( \omega((\omega_i^{(j)})^{(n)}) = \omega_i^{(j)}. \)

**Lemma 2.2.** If \((H_2)\) holds, then \( \frac{d\lambda}{d\tau} \bigg|_{\tau=(\omega_i^{(j)})^{(n)}} \neq 0. \)

**Proof.** Differentiating both sides of Eq. \((2.3)\) with respect to \( \tau \) and replacing \( \tau \) by \( (\omega_i^{(j)})^{(n)} \), we have
\[
\text{Re} \frac{d\lambda}{d\tau} \bigg|_{\tau=(\omega_i^{(j)})^{(n)}} = \frac{g'((\omega_i^{(j)})^{(n)})}{\sqrt{\omega_i^{(j)^2} k^2}} \neq 0.
\]
For $a_1$, $a_2$ and $a_3$, we make the following assumption:

$$(H_3)a_3 > 0, a_1 > 0, \quad \text{and} \quad a_3a_2 - a_1 > 0. \quad \Box$$

**Theorem 2.1.** Suppose that $(H_1)$ and $(H_2)$ are satisfied.

(i) Let $a_1^2 < 2a_2k$.

(a) If $g'(z) > 0$ for all $z > 0$, and $H_3$ holds, then the zero solution of system (2.1) is asymptotically stable for $\tau \in [0, \tau_0)$, where $\tau_0 = \min\{(\tau_{i}^{(j)})^{(0)}, i = 0, 1, 2, j = 1, 2, 3\}$.

(b) If $z_1^*, z_2^*$ are positive zeros of $g'(z)$, and $g(z_1^*) > 0, g(z_2^*) < 0$, then as $\tau$ increases, a number of stability switch may occur, and eventually the system (2.1) becomes unstable.

(2) Let $a_1^2 > 2a_2k$.

(a) If $g'(z) > 0$ for all $z > 0$ and $H_3$ holds, then the zero solution of system (2.1) is asymptotically stable for all $\tau > 0$.

(b) If $z_1^*, z_2^*$ are positive zeros of $g'(z)$, and $g(z_1^*) > 0, g(z_2^*) < 0$, then as $\tau$ increases, a number of stability switch may occur, and eventually the system (2.1) becomes unstable.

(3) The system (2.1) undergoes Hopf bifurcation when $(\tau_{i}^{(j)})^{(n)} (i = 0, 1, 2; j = 1, 2, 3; n = 1, 2, \ldots)$.

**Proof.** When $\tau = 0$, Eq. (2.3) becomes

$$x^3 + a_1x^2 + a_2x + a_1 = 0. \quad (2.9)$$

(i) By the Routh–Hurwitz criterion, all the roots of Eq. (2.9) have negative real parts if and only if $(H_3)$ is satisfied. Hence, all roots of Eq. (2.3) with $\tau = 0$ have negative real parts. From the (1) in Lemma 2.1, we know that all the roots of Eq. (2.3) have negative real parts when $\tau \in [0, \tau_0)$. Applying the Lemma 2.2 we have that Eq. (2.3) has at least a couple of roots with positive real parts when $\tau > \tau_0$. Hence the first conclusion of (1) follows.

(ii) Similarly, Eq. (2.3) has no root with zero real part for all $\tau > 0$. Then all roots of Eq. (2.3) have negative real parts for all $\tau > 0$. Hence, the first conclusion of (2) follows.

(iii) The second conclusions of (1), (2) follow from Lemmas 2.1 and 2.2, and the Theorem 4.1 (p. 83) of Kuang [8].

(iv) The conclusion of (3) is obtained by using the Hopf bifurcation theorem for functional differential equations [9, Theorem 1.1, p. 246–247]. \( \Box \)

3. Direction and stability of the Hopf bifurcation

In Section 2 we obtained some conditions which guarantee that the system (2.1) undergoes the Hopf bifurcation at a sequence values of $\tau$. In this section, we shall study the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions. The method we used is based on the normal form theory and the center manifold theorem introduced by Hassard et al. [10].

For convenience, let us take $y_i(t) = y_i(\tau t), o_0 = o_0^{(j)}$ and $\tau = \tau_* + \nu$, where $\tau_* = (\tau_{i}^{(j)})^{(n)}$ for fixed $n$, and $\nu \in R$. Then system (2.1) becomes

$$\begin{align*}
\dot{y}_1(t) &= (\tau_* + \nu)y_2(t), \\
\dot{y}_2(t) &= (\tau_* + \nu)y_3(t), \\
\dot{y}_3(t) &= (\tau_* + \nu)(-a_1 + k)y_1(t) - a_2y_2(t) - a_3y_3(t) - ky_1(t - \tau) + \frac{\nu^0}{\nu}y_1^2 + \frac{\nu^0}{\nu}y_2^2 + \mathcal{O}(|y|^4).
\end{align*}$$

(3.1)

In order to study the direction of the Hopf bifurcation and stability of the bifurcating periodic solutions, we need to assume that $f \in C^3$. Choosing the phase space as

$$C = C([-1, 0], \mathbb{R}^3).$$

For $\phi \in C$, let

$$L_\nu \phi = B_1\phi(0) + B_2\phi(-1),$$

where

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\[ B_1 = (\tau_s + \nu) \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 + k & -a_2 & -a_3 \end{pmatrix}, \]

\[ B_2 = (\tau_s + \nu) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix}, \]

and

\[ \tilde{F}(v, \phi) = \begin{pmatrix} 0 \\ 0 \\ (\tau_s + \nu) \left[ \frac{1}{2} \dot{\phi}^4(0) + \frac{1}{2} \ddot{\phi}^3(0) + \mathcal{O}(\|\phi\|^4) \right] \end{pmatrix}. \]

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions

\[ \eta(\theta, \nu) : [-1, 0] \to \mathbb{R}^3, \]

such that

\[ L_v \phi = \int_{-1}^{0} d\eta(\theta, \nu) \phi(\theta). \]

In fact, we choose

\[ \eta(\theta, \nu) = \begin{cases} -B_2, & \theta = -1, \\ 0, & \theta \in (-1, 0), \\ B_1, & \theta = 0, \end{cases} \]

the last equation is satisfied.

For \( \phi \in C([-1, 0], \mathbb{R}^3) \), define

\[ A(v)\phi = \begin{cases} \frac{\partial \phi(\theta)}{\partial \theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(t, \nu) \phi(t), & \theta = 0, \end{cases} \]

and

\[ R(v)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ \tilde{F}(v, \phi), & \theta = 0. \end{cases} \]

Hence

\[ \dot{u}_t = A(v)u_t + R(v)u_t, \quad (3.2) \]

where

\[ u = (u_1, u_2, u_3)^T, u_t = u(t + \theta), \quad \theta \in [-1, 0]. \]

For \( \psi \in C^1([0, 1], \mathbb{R}^3) \) define

\[ A^* \psi(s) = \begin{cases} -\frac{\partial \psi(\theta)}{\partial \theta}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta(t, \nu_0) \psi(-t), & s = 0. \end{cases} \]

For \( \phi \in C([-\tau, 0], C^3) \) and \( \psi \in C([0, 1], (C^3)^*) \), define the bilinear form

\[ \langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \psi(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi, \]

where \( \eta(0) = \eta(\theta, 0) \), then \( A = A(0) \) and \( A^* \) are adjoins operators.

By the results of above, \( \pm i\omega_0 \xi \) are eigenvalues of \( A(0) \), thus they are also eigenvalues of \( A^* \).

By direct computation, we obtain that \( q(\theta) = q_0 e^{i\omega_0 \xi \theta} \), with
\[
g_0 = \begin{pmatrix}
1 \\
 i\omega_0 \\
 (i\omega_0)^2
\end{pmatrix},
\]

is the eigenvector of \(A\) corresponding to \(i\omega_0r_\ast\), and \(q'(s) = q_0^e e^{i\omega_0 r_\ast t}\) with
\[
q_0^e = K \begin{pmatrix}
(-i\omega_0)^2 + a_3(-i\omega_0) + a_2 \\
i\omega_0 + a_3 \\
1
\end{pmatrix}^T
\]
is the eigenvector of \(A^*\) corresponding to \(-i\omega_0r_\ast\), where
\[K = [a_0^2 + a_2 + ke^{-i\omega_0}]^{-1}.
\]
Moreover, \(\langle q^*, q \rangle = 1\), and \(\langle q^*, \bar{q} \rangle = 0\).

Using the same notation as in \([10]\), we compute the coordinates to describe the center manifold \(C_0\) at \(v = 0\). Let \(u_i\) be the solution of Eq. (3.2) when \(v = 0\). Define
\[
z(t) = \langle q^*, u_i \rangle, \\
W(t, \theta) = u_i(\theta) - 2\text{Re}\{z(t)\bar{q}(\theta)\}.
\]
On the center manifold \(C_0\), we have
\[
W(t, \theta) = W(z(t), \bar{z}(t), \theta),
\]
where
\[
W(z, \bar{z}, \theta) = W_{20}(\theta)z_2^2 + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\bar{z}_2^2 + W_{00}(\theta) + \cdots,
\]
\(z\) and \(\bar{z}\) are local coordinates for center manifold \(C_0\) in the direction of \(q^*\) and \(\bar{q}^*\). Note that \(W\) is real if \(u_i\) is real. We only consider real solutions. For solution \(u_i \in C_0\) of (3.2), since \(v = 0\),
\[
\dot{z} = i\tau,\omega_0z + \langle q^*(\theta), \tilde{F}(W + 2\text{Re}\{z(t)\bar{q}(\theta)\}) \rangle
\]
\[
= i\tau,\omega_0z + \bar{q}^*(0)\tilde{F}(W(z, \bar{z}, 0) + 2\text{Re}\{z(t)\bar{q}(\theta)\}) \overset{\text{def}}{=} i\tau,\omega_0z + \bar{q}^*(0)\tilde{F}_0(z, \bar{z}).
\]
We rewrite this as
\[
\dot{z}(t) = i\tau,\omega_0z(t) + g(z, \bar{z}),
\]
where
\[
g(z, \bar{z}) = \bar{q}^*(0)\tilde{F}(W(z, \bar{z}, 0) + 2\text{Re}\{z(t)\bar{q}(\theta)\}) = g_{20}z_2^2 + g_{11}z\bar{z} + g_{02}\bar{z}_2^2 + g_{00} + \cdots.
\]
By (3.2) and (3.3), we have
\[
\dot{W} = u_i - \dot{\bar{q}} - \dot{g} = \begin{cases}
AW - 2\text{Re}\{\bar{q}^*(0)\tilde{F}_0q(\theta)\}, & \theta \in [-1, 0), \\
AW - 2\text{Re}\{\bar{q}^*(0)\tilde{F}_0q(\theta)\} + \tilde{F}_0, & \theta = 0,
\end{cases}
\]
where
\[
H(z, \bar{z}, \theta) = H_{20}(\theta)z_2^2 + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\bar{z}_2^2 + \cdots
\]
(3.4)
Expanding the above series and comparing the coefficients, we obtain
\[
(A - 2i\tau,\omega_0)W_{20}(\theta) = -H_{20}(\theta)
\]
\[
AW_{11}(\theta) = -H_{11}(\theta)
\]
\[
\cdots
\]
(3.5)
Denote the \(j\)th element of \(q(0)\) by \(q_j\), and the \(j\)th element of \(W(z, \bar{z}, \theta)\) by
\[
W_j(z, \bar{z}, \theta) = W_{20}(\theta)z_2^2 + W_{11}(\theta)z\bar{z} + \cdots
\]
Then it follows that

\[ x_i(t) = W'(z, \bar{z}, 0) + zq_i + \bar{z}q_i, \quad i = 1, 2, \ldots, n. \]

Hence

\[ \tilde{F}(0, W + zq + \bar{z}q) = \tau \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \tilde{f}''(0)[q_i^2z^2 + 2q_iq_{i1}z\bar{z} + \bar{q}_i^2\bar{z}^2] + \frac{1}{2} \tilde{f}''(0)(W_{10}(0)q_1 + 2W_{11}(0)q_1) + \frac{1}{2} \tilde{f}''(0)[q_i^2q_{i1}z^2 + \cdots] \end{pmatrix}. \]

Notice that

---

Fig. 1. Figure of system (4.1) in \( t-x(1) \) plane. \( a = -0.6, b = 0.17, c = 2 \).

Fig. 2. Figure of system (4.1) in \( t-x(2) \) plane. \( a = -0.6, b = 0.17, c = 2 \).
we have $g_{20} = g_{11} = g_{02} = \tau \bar{K} \dddot{\bar{q}}(0)$, $g_{21} = \tau \bar{K} [f''(0)(2W_{11}^{4}(0) + W_{20}^{0}(0)) + \dddot{\bar{q}}(0)]$.

We still need to compute $W_{20}(\theta)$.

For $\theta \in [-1,0)$, we have

$$H(z, z, \theta) = 2 \Re \{\bar{q}'(0)\dot{F}_0 q(\theta)\} = -gq(\theta) - \dot{\bar{q}}(\theta)$$

$$= -\left(\frac{g_{20}}{2} z_x^2 + g_{11} z_{\bar{x}}^2 + g_{02} 2 z_x \dot{z}_x + \cdots \right)q(\theta) - \left(\frac{g_{20}}{2} z_x^2 + g_{11} z_{\bar{x}}^2 + g_{02} 2 z_x \dot{z}_x + \cdots \right)\dot{q}(\theta).$$

Comparing the coefficients with (3.4) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - g_{02}(\theta) \quad \text{and} \quad H_{11}(\theta) = -g_{11}q(\theta) - g_{11}\dot{q}(\theta).$$

It follows from the definition of $W$ that

$$\dot{W}_{20}(\theta) = 2i \pi \omega_b W_{20}(\theta) - g_{20}q(\theta)e^{i\omega_b \theta} - g_{02}\dot{q}(\theta)e^{-i\omega_b \theta}.$$

Fig. 3. Figure of system (4.1) in $t-x(3)$ plane. $a = -0.6$, $b = 0.17$, $c = 2$.

Fig. 4. System (4.1) has a chaotic attractor. $a = -0.6$, $b = 0.17$, $c = 2$. 

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Solving for this, we obtain
\[ W_{20}(\theta) = \frac{g_{20}}{i\tau_0 \omega_0} q(0) e^{i\tau_0 \omega_0 \theta} - \frac{g_{20}}{3i\tau_0 \omega_0} \bar{q}(0) e^{-i\tau_0 \omega_0 \theta} + E_1 e^{i\tau_0 \omega_0 \theta}, \]
and similarly
\[ W_{11}(\theta) = \frac{g_{11}}{i\tau_0 \omega_0} q(0) e^{i\tau_0 \omega_0 \theta} - \frac{g_{11}}{i\tau_0 \omega_0} \bar{q}(0) e^{-i\tau_0 \omega_0 \theta} + E_2, \]
where \( E_1 \) and \( E_2 \) are both \( n \)-dimensional vectors, and can be determined by setting \( \theta = 0 \) in \( H \). In fact, since

Fig. 5. Figure of system (4.2) in \( t-x(1) \) plane. \( a = -0.6, b = 0.17, c = 2, k = -10, \) delay = 0.15.

Fig. 6. Figure of system (4.2) in \( t-x(2) \) plane. \( a = -0.6, b = 0.17, c = 2, k = -10, \) delay = 0.15.
\[ H(z, z, 0) = -2 \text{Re}\{q^*(0)\hat{F}_0 q(0)\} + \hat{F}_0, \]

we have

\[ H_{20} = -g_{20} q(0) - \g_{20} \dddot{q}(0) + \tau \begin{pmatrix} 0 \\ 0 \\ \dddot{q}(0) q_1^2 \end{pmatrix}, \]

\[ H_{11} = -g_{11} q(0) - \g_{11} \dddot{q}(0) + \begin{pmatrix} 0 \\ 0 \\ \dddot{q}(0) q_1 \end{pmatrix}, \]

we get

![Figure 7](image1.png)

**Fig. 7.** Figure of system (4.2) in \( t-x(3) \) plane. \( a = -0.6, b = 0.17, c = 2, k = -10, \) delay = 0.15.

![Figure 8](image2.png)

**Fig. 8.** The chaos vanished, periodic solutions are asymptotically stable.
Solving this we can obtain \( E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^\top \). It is clearly that \( E_1^{(3)} = 0 \). Similarly, we can get \( E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^\top \) from

\[
\begin{pmatrix}
2i\omega_0 & 1 & 0 \\
0 & 2i\omega_0 & 1 \\
-a_1 + k - ke^{-2\omega_0\tau} & -a_2 & -a_3
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\hat{r}(0)\bar{q}_1
\end{pmatrix}.
\]

Fig. 9. Figure of system (4.2) in \( t-x(1) \) plane. \( a = -0.6, b = 0.17, c = 2, k = -10, \) delay = 0.18.

Fig. 10. Figure of system (4.2) in \( t-x(2) \) plane. \( a = -0.6, b = 0.17, c = 2, k = -10, \) delay = 0.18.
and $E^{(3)}_2 = 0$. Based on the analysis, we can obtain:

$$g_{20} = g_{11} = g_{02} = \tau \hat{K} \hat{f}''(0), \quad g_{21} = \frac{\tau}{2} [2k \hat{f}'''(0) + \hat{f}'''(0)],$$

where

$$K = [\omega_0^2 + a_2 + k e^{-i\omega_0 \tau}]^{-1}.$$ 

Thus, we can compute the following quantities:

$$c_1(0) = \frac{i}{2\omega_0} \left( g_{20} g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2},$$

$$\mu_2 = - \frac{\text{Re} c_1(0)}{\text{Re} \lambda_2(\tau_0)},$$

$$\beta_2 = 2 \text{Re} c_1(0).$$

![Fig. 11. Figure of system (4.2) in $t-x(3)$ plane. $a = -0.6$, $b = 0.17$, $c = 2$, $k = -10$, delay = 0.18.](image1)

![Fig. 12. The equilibrium ($-3.43, 0, 0$) is asymptotically stable.](image2)
We know that (see [10]) $\mu_2$ determines the directions of the Hopf bifurcation: if $\mu_2 > 0 (<0)$, then the Hopf bifurcations are supercritical (subcritical) and the bifurcation periodic solutions exist for $\mu > \mu_0 (<\mu_0)$; $\beta_2$ determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are orbitally stable (unstable) if $\beta_2 < 0 (>0)$.

Fig. 13. Figure of system (4.2) in $t-x(1)$ plane. $a = -0.6, b = 0.17, c = 2, k = -10$, delay = 0.22.

Fig. 14. Figure of system (4.2) in $t-x(2)$ plane. $a = -0.6, b = 0.17, c = 2, k = -10$, delay = 0.22.
4. Application to chaotic control

Consider the following Jerk model:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= x_3(t), \\
\dot{x}_3(t) &= ax_3 - x_2 + bx_2^2 - c.
\end{align*}
\]  

(4.1)

If \( a = -0.6, \ b = 0.17, \ c = 2 \), then the system has a chaotic attractor [11]. See Figs. 1–4.

We add a time-delayed force \( k(x_1(t) - x_1(t - \tau)) \) to the third equation of system (4.1), then we have the following delayed feedback control system:

![Periodic solutions bifurcate from the equilibrium again and the bifurcating periodic solutions are asymptotically stable.](image1)

Fig. 15. Figure of system (4.2) in \( t-x(3) \) plane. \( a = -0.6, \ b = 0.17, \ c = 2, \ k = -10, \) delay = 0.22.

![Periodic solutions bifurcate from the equilibrium again and the bifurcating periodic solutions are asymptotically stable.](image2)

Fig. 16. Periodic solutions bifurcate from the equilibrium again and the bifurcating periodic solutions are asymptotically stable.
\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= x_3(t), \\
\dot{x}_3(t) &= ax_3 - x_2 + bx_1^2 - c + k(x_1(t) - x_1(t - \tau)).
\end{align*}
\] (4.2)

We investigate the effect of delay on the dynamic behavior of system (4.2). The equilibrium of Eq. (4.1) are \((\pm \sqrt{\frac{c}{b}}, 0, 0)\). Following the discussion of Section 2, we choose \(k = -10, \tau = 0.15\). In this case, a periodic solution bifurcates from the equilibrium and the bifurcating periodic solution is asymptotically stable, the chaos vanished. See Figs. 5–8.

In Figs. 9–12, \(\tau = 0.18\). They show that the equilibrium \((-3.43, 0, 0)\) is asymptotically stable and the chaos vanished.

Fig. 17. Figure of system (4.2) in \(t-x(1)\) plane. \(a = -0.6, b = 0.17, c = 2, k = -10, \text{delay} = 0.34\).

Fig. 18. Figure of system (4.2) in \(t-x(2)\) plane. \(a = -0.6, b = 0.17, c = 2, k = -10, \text{delay} = 0.34\).
In Figs. 13–16, \( s = 0.22 \). A periodic solution bifurcates from the equilibrium again and the bifurcating periodic solution is asymptotically stable.

In Figs. 17–20, \( s = 0.34 \). A chaotic attractor appears again.

References


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