Double Hopf bifurcation for van der Pol-Duffing oscillator with parametric delay feedback control

Suqi Ma a, Qishao Lu b, Zhaosheng Feng c,∗

a Department of Mathematics, China Agricultural University, Beijing 100083, China
b School of Science, Beijing University of Aeronautics and Astronautics, Beijing 100083, China
c Department of Mathematics, University of Texas-Pan American, Edinburg, TX 78541, USA

Received 9 April 2007
Available online 7 June 2007
Submitted by Goong Chen

Abstract
The stability and bifurcation of a van der Pol-Duffing oscillator with the delay feedback are investigated, in which the strength of feedback control is a nonlinear function of delay. A geometrical method in conjunction with an analytical method is developed to identify the critical values for stability switches and Hopf bifurcations. The Hopf bifurcation curves and multi-stable regions are obtained as two parameters vary. Some weak resonant and non-resonant double Hopf bifurcation phenomena are observed due to the vanishing of the real parts of two pairs of characteristic roots on the margins of the “death island” regions simultaneously. By applying the center manifold theory, the normal forms near the double Hopf bifurcation points, as well as classifications of local dynamics are analyzed. Furthermore, some quasi-periodic and chaotic motions are verified in both theoretical and numerical ways. © 2007 Elsevier Inc. All rights reserved.

Keywords: Van der Pol-Duffing oscillator; Chaos; Double Hopf bifurcation; Stability; Phase portrait; Center manifold

1. Introduction
The effects of delay are popular in dynamical systems due to the finite propagation speed of signals or the finite time of processing or reacting. In the study of delayed systems, many delay factors appear in state variables and some of them appear in parameters. In recent years, delay feedback control is widely applied in mechanical and electronic facilities. In the present paper, we consider a van der Pol-Duffing oscillator controlled by the parametric delay feedback

\[ \ddot{x}(t) + \omega_0^2 x(t) - [b - \gamma x^2(t)] \dot{x}(t) + \beta x^3(t) = A[e^{-p\tau} x(t - \tau) - x(t)], \]

(1.1)

where \( \omega_0, b, \gamma, A, p \) are positive real constants, and \( \tau \geq 0 \) is the time delay. Note that the strength of feedback control takes the form of \( Ae^{-p\tau} \), a function exponentially decreasing with the time delay. This implies that the feedback effect

✩ The work is supported by UTPA Faculty Research Council Grant 119100 and NSF of China (Nos. 10432010 and 10572011), and partially supported by the Doctoral Grant 2006064 originated from the Chinese Agricultural University.

* Corresponding author. Fax: +1(956) 384 5091.
E-mail address: zsfeng@utpa.edu (Z. Feng).
of the past state is fading with the time $t$. Mathematically, the bigger both $\tau$ and $p$ are, the smaller the factor $Ae^{-p\tau}$ is. Hence, $p$ is called the decay rate. Equation (1.1) behaves like a van der Pol-Duffing oscillator as $\tau$ or $p$ becomes sufficiently large, which means the vanishing of the delay effect. Apparently, the strength of feedback control $Ae^{-p\tau}$ becomes the delay-independent constant $A$ as $p = 0$, then Eq. (1.1) becomes the van der Pol-Duffing equation with a regular delay. Therefore, Eq. (1.1) appears to be a more general model and is of much interest in several areas such as biology and engineering.

Delay differential equations (DDEs) with delay-dependent parameters can be referred to some ‘stage-structured’ models in biology (see [1–5]). In one of our recent works [6] we discussed the stability and the Hopf bifurcation of a population model with delay-dependent parameters controlled by the parametric delay feedback. Many research works on dynamics of DDEs are concerned with delay-independent parameters [7–11] etc. and much attention [12–14] has been carried out on the double Hopf bifurcation and complex dynamics of delayed oscillators with time-independent parameters. The center manifold method was adopted to reduce DDEs into finite-dimensional systems [15–21], and the bifurcation analysis of DDEs could be achieved for the reduced systems.

In this paper, a geometrical method for studying stability of Eq. (1.1) with delay-dependent parameters is developed on a two-parameter plane. Beretta and Kuang [5] had presented an elegant geometrical criterion with the delay $\tau$ as a single bifurcation parameter. In order to get a deeper insight into the higher-dimensional bifurcation analysis of Eq. (1.1), we may apply and extend the geometrical method to the case of two bifurcation parameters and consider the effect of the additional parameter $p$ for stability switching at the critical values of delay. Complex dynamical behaviors of Eq. (1.1), including Hopf bifurcations and multi-stable regions, double Hopf bifurcation, quasi-periodic and chaotic motions, are analyzed theoretically and numerically. In particular, the double Hopf bifurcations are induced if two pairs of imaginary roots appear simultaneously on the margins of the “death island” regions.

The rest of this paper is organized as follows. In Section 2, we present a two-parameter geometrical criterion to the stability and the Hopf bifurcations of Eq. (1.1). Some weak resonant and non-resonant double Hopf bifurcations are analyzed and the necessary conditions for double Hopf bifurcations are discussed. In Section 3, computations of normal forms and universal unfoldings at the double Hopf bifurcation points are carried out, and local classification in the neighborhood of double Hopf points is undertaken. At the end of this section, numerical simulations are done to reveal dynamical behaviors near double Hopf bifurcation points, such as quasi-periodical solutions and chaos. The Neimark–Sacker bifurcation is also detected. In Section 4, we present a brief conclusion.

2. Stability analysis

In this section, we divide our arguments into two parts. We first investigate the stability-switches and the Hopf bifurcation for system (1.1), where the delay $\tau$ and the decay rate $p$ are chosen as bifurcation parameters. Then we study some resonant and non-resonant double Hopf bifurcations of system (1.1).

2.1. Hopf bifurcation

Since Eq. (1.1) has an equilibrium solution $x = 0$, making a linearization of system (1.1) near $x = 0$ leads to the characteristic equation

$$\Delta(\lambda, \tau) = \lambda^2 - b\lambda - Ae^{-(p+\lambda)\tau} + (A + \omega_0^2) = 0. \quad (2.1)$$

Substituting $\lambda = \alpha + i\omega$ (with the assumption $\omega > 0$) into Eq. (2.1) and equating both the real and imaginary parts to zero yields

$$\alpha^2 - \omega^2 - b\alpha - Ae^{-p\tau}e^{-\alpha\tau}\cos(\omega\tau) + A + \omega_0^2 = 0,$$

$$2\alpha\omega - b\omega + Ae^{-p\tau}e^{-\alpha\tau}\sin(\omega\tau) = 0. \quad (2.2)$$

The stability of the trivial solution may change and the Hopf bifurcation may occur when some characteristic roots of Eq. (2.1) have zero real parts. Set $\alpha = 0$, then Eq. (2.2) becomes

$$-\omega^2 + (A + \omega_0^2) = Ae^{-p\tau}\cos(\omega\tau),$$

$$b\omega = Ae^{-p\tau}\sin(\omega\tau). \quad (2.3)$$
It is not easy to express \( (\omega, \tau) \) explicitly from Eq. (2.3) due to the exponential factor \( Ae^{-\rho \tau} \). Thus we introduce a new variable \( \theta = \omega \tau \) to Eq. (2.3) instead of \( (\omega, \tau) \), and we have

\[
\begin{align*}
\sin \theta &= \frac{b \omega}{Ae^{-\rho \tau}}, \\
\cos \theta &= \frac{-\omega^2 + A + \omega_0^2}{Ae^{-\rho \tau}},
\end{align*}
\]

which can be further transformed into

\[
\begin{align*}
p_\pm &= -\frac{\omega_\pm}{\theta} \ln \frac{b \omega_\pm}{A \sin \theta}, \\
t_\pm &= -\frac{1}{p_\pm} \ln \frac{b \omega_\pm}{A \sin \theta},
\end{align*}
\]

with

\[
\omega_\pm = -b \pm \sqrt{b^2 + 4(A + \omega_0^2) \tan^2 \theta}
\]

It can be seen from Eq. (2.4) that \( \sin \theta > 0 \) due to our assumptions. Let \( \theta = \theta_0 + 2n\pi \) \((n = 0, 1, 2, \ldots)\), and substitute it into Eq. (2.6). To ensure the positivity of \( \omega_\pm \) in Eq. (2.6), there is a limit condition for \( \theta_0 \):

\[
\begin{align*}
\theta_0 &\in (0, \pi/2], \quad \text{for } \omega_+ > 0, \\
\theta_0 &\in [\pi/2, \pi), \quad \text{for } \omega_- > 0.
\end{align*}
\]

From (2.6) and (2.7) it is easy to see that when \( \theta_0 = \pi/2 \) we have \( \omega_- = \omega_+ \) in Eq. (2.4) for any \( n \), which leads to \( p_- = p_+ \), \( \tau_\theta = \tau_+ \). Thus, for each \( n \), the curves \( p_-(\theta) \) and \( p_+(\theta) \) will intersect at \( \theta_0 = \pi/2 \) as \( \theta_0 \) varies in \((0, \pi)\).

Discussions for the case of the curves \( \tau_- (\theta) \) and \( \tau_+ (\theta) \) are similar.

For the case of \( \tau = 0 \), the zero solution is an unstable spiral point since \( \alpha = b/2, \omega > 0 \) according to Eq. (2.2).

We discuss the case \( \tau > 0 \) by Eq. (2.5). The numbers of \( \theta \) satisfying Eq. (2.5), continually increases by reducing \( p \) or raising \( \tau \). This variation causes Hopf bifurcations simultaneously. The geometrical chart of the curves \( p = p_\pm \) in Eq. (2.5) affords us an intuitive insight about the frequent occurrence of Hopf bifurcations as shown in Fig. 1, where the red curves denote \( p_+ \) while the blue curves represent \( p_- \) for \( n = 0, 1, 2, 3 \), respectively. The geometrical results can be deduced and summarized as follows:

(1a) If \( p > p_n^{\max} \), the characteristic equation (2.1) has no imaginary roots, thus no Hopf bifurcation occurs. System (1.1) is unstable for all delay \( \tau > 0 \).

(1b) If \( p_{n+1}^{\max} < p < p_n^{\max} \), the total number of pure imaginary roots of Eq. (2.1) is \( 2(n + 1) \) for \( n = 0, 1, 2, \ldots \). This implies that Hopf bifurcations for system (1.1) may appear \( 2(n + 1) \) times as the delay varies.

(1c) If \( p = p_n^{\max} \), the characteristic equation (2.1) has totally \( 2n + 1 \) pure imaginary roots, for \( n = 0, 1, 2, \ldots \). This implies that Hopf bifurcations for system (1.1) may appear \( 2n + 1 \) times as the delay varies.

For each \( n \), the maximal value \( p_n^{\max} \) can be computed by \( \frac{dp_\pm(\theta)}{d\theta} = 0 \). It is obvious that from Eqs. (2.5) and (2.6) we have

\[
\frac{dp_\pm(\theta)}{d\theta} = \frac{p_\pm \tau_\pm \omega_\pm b(1 + \tan^2 \theta)}{\theta \tan \theta (2\omega_\pm \tan \theta + b)} + \frac{\omega_\pm (-b \sin \theta + 2\omega_\pm \cos \theta)}{\theta (2\omega_\pm \sin \theta + b \cos \theta)} - \frac{p_\pm \omega_\pm \tau_\pm \omega_\pm b(1 + \tan^2 \theta)}{\theta^2} = \frac{p_\pm b \omega_\pm - 2\omega_\pm b \sin^2 \theta - b \omega_\pm b \cos^2 \theta - b \omega_\pm \sin \theta \cos \theta + 2\omega_\pm \sin \theta \cos \theta}{\theta \sin \theta (2\omega_\pm \sin \theta + b \cos \theta)}.
\]

Substituting \( p_\pm \) from Eq. (2.5) and \( \omega_\pm \) from Eq. (2.6) into Eq. (2.8) and solving for \( \theta_n^{\max} \) from \( \frac{dp_\pm(\theta)}{d\theta} = 0 \) for each given \( n \), we can derive the value \( p_n^{\max} \) from Eq. (2.5). As shown in Fig. 1, the maximal points \( (\theta_n^{\max}, p_n^{\max}) \) are denoted by dots for \( n = 0, 1, 2, 3 \), respectively. Moreover, the Hopf bifurcation values of delay \( \tau \) can be also found from Eq. (2.5). For the given \( p \), we denote the roots of the equation \( p_\pm(\theta) = p \) by \( \theta_{k,n} \) \((k = 1, 2)\), the corresponding characteristic roots by \( \omega_{k,n} \) and the bifurcation delay values by \( \tau_{k,n} \), respectively.
Consider the stability-switch and Hopf bifurcations of Eq. (1.1). For the given \( \tau \), we suppose that Eq. (2.2) has two distinct solutions \( \alpha(\tau) \pm i\omega(\tau) \). Denoting \( \tau_c = \tau_{k,n} \), \( \omega_c = \omega_{k,n} \) (\( k = 1, 2; n = 0, 1, 2, \ldots \)) for simplicity in what follows, we have \( \alpha(\tau_c) = 0 \) and \( \omega(\tau_c) = \omega_c \). Differentiating equation (2.2) with respect to \( \tau \), we have
\[
\left( A\tau e^{-\rho \tau} \cos(\omega_c \tau) - b \right) \frac{d\alpha}{d\tau} + \left( A\tau e^{-\rho \tau} \sin(\omega_c \tau) - 2\omega_c \right) \frac{d\omega_c}{d\tau} = -A\rho e^{-\rho \tau} \cos(\omega_c \tau) - A\omega_c e^{-\rho \tau} \sin(\omega_c \tau),
\]
\[
-\left( A\tau e^{-\rho \tau} \sin(\omega_c \tau) - 2\omega_c \right) \frac{d\alpha}{d\tau} + \left( A\tau e^{-\rho \tau} \cos(\omega_c \tau) - b \right) \frac{d\omega_c}{d\tau} = A\rho e^{-\rho \tau} \sin(\omega_c \tau) - A\omega_c e^{-\rho \tau} \cos(\omega_c \tau).
\]
(2.9)

Solving for \( \frac{d\alpha}{d\tau} \) from Eq. (2.9) leads to
\[
\left| \frac{d\alpha}{d\tau} \right|_{\tau = \tau_c} = -\frac{F}{(A\tau_c \sin(\omega_c \tau_c) - 2\omega_c e^{\rho \tau_c})^2 + (A\tau_c \cos(\omega_c \tau_c) - be^{\rho \tau_c})^2},
\]
(2.10)
where
\[
F = A \left( -b\omega_c e^{\rho \tau_c} \sin(\omega_c \tau_c) - 2\omega_c p e^{\rho \tau_c} \sin(\omega_c \tau_c) + A\tau_c \rho e^{\rho \tau_c} \cos(\omega_c \tau_c) + 2\omega_c^2 e^{\rho \tau_c} \cos(\omega_c \tau_c) \right),
\]
which, using (2.5), can be rewritten as
\[
F = -\frac{A^2}{b\omega_c} \left( \rho \tau_c b\omega_c - 2\omega_c p \sin^2 \theta - b\omega_c \sin^2 \theta - bp \sin \theta \cos \theta + 2\omega_c^2 \sin \theta \cos \theta \right).
\]
(2.11)

From Eqs. (2.8), (2.10) and (2.11) we have the inequality \( \frac{d\alpha}{d\tau} \cdot \frac{dp}{d\theta} \bigg|_{\tau = \tau_c} < 0 \).

As shown in Fig. 1(a) and Fig. 1(b), due to the uniqueness of \( p_{n}^{\max} \), for each \( n \) we get
\[
\frac{dp}{d\theta} \left\{ \begin{array}{ll}
> 0, & \text{if } \tau = \tau_{1,n} \text{ with } n = 0, 1, 2, \ldots, \\
< 0, & \text{if } \tau = \tau_{2,n} \text{ with } n = 0, 1, 2, \ldots.
\end{array} \right.
\]
(2.12)
Thus, we have
\[
\frac{d\alpha}{d\tau} \left\{ \begin{array}{ll}
< 0, & \text{if } \tau = \tau_{1,n} \text{ for } n = 0, 1, 2, \ldots, \\
> 0, & \text{if } \tau = \tau_{2,n} \text{ for } n = 0, 1, 2, \ldots.
\end{array} \right.
\]
(2.13)
Based on the analysis in the previous work [6], we know that the equilibrium \( x = 0 \) is unstable for the case without the delay (that is, \( \tau = 0 \)). When \( \tau \) increases continuously, the situation remains unchanged until the first critical value \( \tau_{1,0} \) corresponding to a pair of characteristic roots with the real part being zero, and the Hopf bifurcation occurs at \( \tau_{1,0} \).

According to (2.12) and (2.13), \( \frac{d\alpha}{d\tau} \) is always negative on the dashed lines, whereas it is positive on the solid lines in Fig. 2. As \( \tau \) increases to the next critical value \( \tau_{2,0} \), the trivial solution switches from a stable one to an unstable
one again. Thus a stability region is formed in the \((\tau, p)\) plane, as shown in Fig. 2, and the Hopf bifurcation occurs on the boundary of this region. Generally, more stability regions can be formed if \(\tau_{1,j} < \tau_{2,j}\) for \(1 \leq j \leq m, m \in \mathbb{Z}^+\). As shown in Fig. 2, the multiple stability regions are plotted by the shaded ones. These stability regions are also termed “amplitude death regions” or “death islands” in the \((\tau, p)\) plane [7].

2.2. Double Hopf bifurcation

In the last subsection, the multiple stability regions are formed in the \((\tau, p)\) plane, which implies multi-time switches between the stable and unstable states of zero solution. However, when \(p = p_c\), if \(\tau_{1,j} = \tau_{2,j} = \tau_c\) is satisfied for some \(j \geq 1\), taking account of the intersection of the curves \(\tau_+\) and \(\tau_-\) in Eq. (2.5), a double Hopf bifurcation point \((p_c, \tau_c)\) appears. As the values of the parameter \(p\) and the delay \(\tau\) vary, a number of double Hopf bifurcation points arise. Consequently, some resonant and non-resonant double Hopf bifurcation phenomena can be observed. From Eq. (2.5), the condition for the double Hopf bifurcation \(\tau_c = \tau_{1,j} = \tau_{2,j}\) for the given \(j\) determines the following relationship:

\[
\frac{\omega_2}{\theta_{2,j}} = \frac{\omega_1}{\theta_{1,j}},
\]  

(2.14)

From the first equation of (2.4), we have another necessary condition between \(\theta_{1,j}\) and \(\theta_{2,j}\) for the double Hopf bifurcation

\[
\frac{\omega_1}{\sin\theta_{1,j}} = \frac{\omega_2}{\sin\theta_{2,j}}.
\]  

(2.15)

The quality \(\tau_{1,j} = \tau_{2,j}\) means that the linearized system around the trivial equilibrium has two pairs of purely imaginary eigenvalues \(\pm i\omega_1\) and \(\pm i\omega_2\) simultaneously. If we assume that

\[
\omega_1 : \omega_2 = k_1 : k_2,
\]  

(2.16)

then a possible double Hopf bifurcation with the ratio \(k_1 : k_2\) appears. If \(k_1, k_2 \in \mathbb{Z}^+\), it is called a double Hopf bifurcation point with the \(k_1 : k_2\) resonant; otherwise, it is called a non-resonant double Hopf bifurcation point.

From (2.14) to (2.16), we conclude that the necessary conditions for the \(k_1 : k_2\) double Hopf bifurcation are

\[
\begin{align*}
\theta_{1,j} & = \frac{k_1}{k_2}, \\
\theta_{2,j} & = \frac{k_2}{k_2}, \\
\sin\theta_{1,j} & = \frac{\theta_{2,j}}{\sin\theta_{2,j}}.
\end{align*}
\]  

(2.17)
Table 1

<table>
<thead>
<tr>
<th>$\tau_c$</th>
<th>$\theta_{1,j}$</th>
<th>$\theta_{2,j}$</th>
<th>$k_1:k_2$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.098667428/b</td>
<td>0.3884049526 + 2\pi</td>
<td>2.612268373 + 2\pi</td>
<td>3:4</td>
<td>1</td>
</tr>
<tr>
<td>1.478078365/b</td>
<td>0.6137205354 + 2\pi</td>
<td>2.337946996 + 2\pi</td>
<td>4:5</td>
<td>1</td>
</tr>
<tr>
<td>1.657819638/b</td>
<td>0.7599010934 + 2\pi</td>
<td>2.168518374 + 2\pi</td>
<td>5:6</td>
<td>1</td>
</tr>
<tr>
<td>1.757383599/b</td>
<td>0.8622829092 + 2\pi</td>
<td>2.053194278 + 2\pi</td>
<td>6:7</td>
<td>1</td>
</tr>
<tr>
<td>0.760125052/b</td>
<td>0.2611186776 + 4\pi</td>
<td>2.826616536 + 4\pi</td>
<td>5:6</td>
<td>2</td>
</tr>
<tr>
<td>1.146883417/b</td>
<td>0.4457921522 + 4\pi</td>
<td>2.614485947 + 4\pi</td>
<td>6:7</td>
<td>2</td>
</tr>
<tr>
<td>0.6282545121/b</td>
<td>0.1899799519 + 2\pi</td>
<td>2.871252794 + 2\pi</td>
<td>1: $\sqrt{2}$</td>
<td>1</td>
</tr>
<tr>
<td>1.573303779/b</td>
<td>0.686483193 + 2\pi</td>
<td>2.252874472 + 2\pi</td>
<td>2: $\sqrt{2}$</td>
<td>1</td>
</tr>
<tr>
<td>1.055187749/b</td>
<td>0.3671310591 + 2\pi</td>
<td>2.639150374 + 2\pi</td>
<td>$\sqrt{2}$:3</td>
<td>1</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>$b$</th>
<th>$\omega_0$</th>
<th>$A$</th>
<th>$p_c$</th>
<th>$\tau_c$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_1: \omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.27</td>
<td>0.8</td>
<td>3.129993919</td>
<td>0.242056130</td>
<td>4.069138624</td>
<td>1.639558362</td>
<td>2.186077816</td>
<td>3:4</td>
</tr>
<tr>
<td>0.27</td>
<td>0.8</td>
<td>1.430094705</td>
<td>0.161530833</td>
<td>5.47436431</td>
<td>1.259855107</td>
<td>1.574818884</td>
<td>4:5</td>
</tr>
<tr>
<td>0.27</td>
<td>0.8</td>
<td>24.72443217</td>
<td>0.5848121765</td>
<td>2.815277959</td>
<td>4.556384654</td>
<td>5.467661585</td>
<td>5:6</td>
</tr>
<tr>
<td>0.27</td>
<td>0.8</td>
<td>11.00507934</td>
<td>0.437364220</td>
<td>2.326868561</td>
<td>2.781921320</td>
<td>3.934230859</td>
<td>1: $\sqrt{2}$</td>
</tr>
<tr>
<td>0.27</td>
<td>0.8</td>
<td>1.184732003</td>
<td>0.144807250</td>
<td>5.82705103</td>
<td>1.196087623</td>
<td>1.464902182</td>
<td>2: $\sqrt{2}$</td>
</tr>
<tr>
<td>0.27</td>
<td>0.8</td>
<td>3.450421545</td>
<td>0.2537330783</td>
<td>3.908102771</td>
<td>1.701673870</td>
<td>2.283035070</td>
<td>$\sqrt{2}$:3</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

The double Hopf bifurcation point $(p_c, \tau_c)$ is given by

\[
\begin{align*}
p_c &= -\frac{\omega_1}{\theta_{1,j}} \ln \frac{b\omega_1}{A \sin \theta_{1,j}}, \\
\tau_c &= -\frac{1}{p_c} \ln \frac{b\omega_1}{A \sin \theta_{1,j}}.
\end{align*}
\]

Now we can compute the values of $\theta_{1,j}$ and $\theta_{2,j}$ for the $k_1 : k_2$ double Hopf bifurcation by using Eq. (2.17). The critical values $\tau_c$ can be found from (2.18) as the function only depends on the parameter $b$. For example, here we just consider $j = 1, 2$ and several sets of $k_1 : k_2$. The values of $\theta_{1,j}$, $\theta_{2,j}$ and the critical values $\tau_c$ are illustrated in Table 1.

If we choose the parameters $b = 0.27$, $\omega_0 = 0.8$, the value of parameter $A$ and the corresponding bifurcation parameter pairs $(p_c, \tau_c)$ for double Hopf bifurcations can be obtained and listed in Table 2.

3. Normal form for double Hopf bifurcations

In this section, by means of the center manifold theory and the normal form technique, we study the bifurcation direction and the stability of bifurcating periodic solution at the double Hopf bifurcation point of Eq. (1.1) for $p = p_c$, $\tau = \tau_c$ as presented in the preceding section. For calculational purposes, we first re-scale Eq. (1.1) by $x \rightarrow \sqrt{e}x$, then it becomes

\[
\ddot{x} + \omega_0^2 x - (b - \varepsilon \gamma x^2) \dot{x} + \varepsilon \beta x^3 = A e^{-\varepsilon \tau} x(t - \tau) - Ax.
\]

Setting $p = p_c + \varepsilon p_c$ and $\tau = \tau_c + \varepsilon \tau_c$, Eq. (3.1) is equivalent to

\[
\begin{align*}
\dot{x}(t) &= y, \\
\dot{y}(t) &= -\omega_0^2 x + b y - A x + A e^{-p_c \tau_c} x(t - \tau_c) + A e^{-p_c \tau_c} [x(t - \tau_c - \varepsilon \tau_c) - x(t - \tau_c)] \\
&\quad + \varepsilon (-\gamma x^2 y + A e^{-p_c \tau_c} (-p_c \tau_c - p_c \tau_c)x(t - \tau_c) - \beta x^3) + O(\varepsilon^2).
\end{align*}
\]
Choose the phase space $C = C([-\tau, 0], \mathbb{R}^2)$ as the Banach space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^2$ with the supremum norm. For $\phi \in C$, we define

$$L(0)\phi = \int_{-\tau}^{0} \left[ d\eta(\theta) \right] \phi(\theta),$$

where $\eta: [-\tau, 0] \to \mathbb{R}^2 \times \mathbb{R}^2$ is a real-valued function of bounded variation in $[-\tau, 0]$ with

$$d\eta(\theta) = \begin{bmatrix} 0 & \delta(\theta) \\ -\left(\omega_0^2 + A\right)\delta(\theta) + Ae^{-\nu_c\tau} \delta(\theta + \tau_c) & b\delta(\theta) \end{bmatrix} d\theta. \quad (3.3)$$

Further, we define

$$L(p_\varepsilon, \tau_\varepsilon)\phi = \int_{-\tau}^{0} \left[ d\eta_1(\theta, p_\varepsilon, \tau_\varepsilon) + d\eta_2(\theta, \tau_\varepsilon) \right] \phi(\theta),$$

where

$$d\eta_1(\theta, p_\varepsilon, \tau_\varepsilon) = \begin{bmatrix} 0 & \delta(\theta + \tau_c) \\ -\varepsilon Ae^{-\nu_c\tau_c}(p_\varepsilon \tau_c + \tau_\varepsilon \nu_c) \delta(\theta + \tau_c) & 0 \end{bmatrix} d\theta,$$

and

$$d\eta_2(\theta, \tau_\varepsilon) = \begin{bmatrix} 0 & \delta(\theta + \tau_c + \varepsilon \tau_\varepsilon) - \delta(\theta + \tau_c) \\ Ae^{-\nu_c\tau_c}(\delta(\theta + \tau_c + \varepsilon \tau_\varepsilon) - \delta(\theta + \tau_c)) & 0 \end{bmatrix} d\theta.$$

For $\phi \in C$, the linear operator defined by system (3.2) generates an infinitesimal generator of the semi-flow of bounded linear operators with

$$A(0)\phi = \begin{cases} \frac{\partial \phi}{\partial \theta}, & \theta \in [-\tau, 0), \\ L(0)\phi, & \theta = 0. \end{cases}$$

and

$$A(p_\varepsilon, \tau_\varepsilon)\phi = \begin{cases} 0, & \theta \in [-\tau, 0), \\ L(p_\varepsilon, \tau_\varepsilon)\phi, & \theta = 0. \end{cases}$$

Set

$$Q(\phi) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ F(\phi), & \theta = 0, \end{cases}$$

with

$$F(\phi) = \begin{bmatrix} 0 \\ -\gamma \phi_1^2(0)\phi_2(0) - \beta \phi_1^3(0) \end{bmatrix}.$$

Then system (3.2) can be re-expressed as an operator differential equation

$$u'(t) = A(0)u_t + A(p_\varepsilon, \tau_\varepsilon)u_t + \varepsilon Qu_t, \quad (3.4)$$

where $u = (x, y)^T$ and $u_t = u(t + \theta)$, for $-\tau \leq \theta < 0$. For $\psi \in C^\ast = C([0, \tau], \mathbb{R}^2)$, the adjoint operators $A^\ast(0)$ and $A^\ast(p_\varepsilon, \tau_\varepsilon)$ of $A(0)$ and $A(p_\varepsilon, \tau_\varepsilon)$, respectively, are given by

$$A^\ast(0)\psi(s) = \begin{cases} \frac{\partial \psi}{\partial s}, & 0 < s \leq \tau, \\ \int_{-\tau}^{0} d\eta^T(s)\psi(-s), & s = 0, \end{cases}$$
From the discussion given in Section 2, we know that for \( p \) with \( \text{Re}(p) < 0 \) and \( 1000 \) negative real parts. We then designate the linear operator \( L(\cdot) \) in order to compute \( \Phi \). Thus, the explicit expressions for \( L(\cdot) \) associated with the imaginary characteristic roots. We decompose \( C \) as the 4-dimensional center subspace spanned by the basic vectors of the linear operator \( L(0) \) associated with the imaginary characteristic roots. We decompose \( C = P_A \oplus Q_A \), where \( Q_A \) is the complement subspace of \( P_A \). Referring to \([18,19]\), Eq. (3.1) can be rewritten as an ordinary differential equation in the Banach space \( C \) of functions bounded and continuous on \( [-\tau, 0) \) with a possible jump discontinuity at 0. Elements of \( C \) are of the form \( \phi + X_0 \alpha \), where \( \phi \in C, \alpha \in R^n \) and \( X_0(\theta) = 0 \) for \( \theta \in [-\tau, 0) \) and \( X_0(0) = I \). Let \( \tau : C \to P_A \) be a continuous projection defined by \( \tau(\phi + X_0 \alpha) = \tau[(\Psi, \phi) + (\Psi(\tau) \alpha)] \).

We suppose that the bases of \( P_A \) and \( P_A^* \), respectively, are

\[
\Phi(\theta) = (q_1(\theta), q_1(\theta), q_2(\theta), q_2(\theta)), \quad \Psi(s) = (q_1^*(s), q_1^*(s), q_2^*(s), q_2^*(s)),
\]

with

\[
\langle q_1^*, q_1 \rangle = 1, \quad \langle q_2^*, q_2 \rangle = 0, \quad \langle q_1^*, q_2 \rangle = 0, \quad \langle q_1^*, q_2 \rangle = 0.
\]

It can be computed directly that

\[
q_1 = \left( \frac{1}{i \omega_1} \right) e^{i \omega_1 \theta}, \quad q_2 = \left( \frac{1}{i \omega_2} \right) e^{i \omega_2 \theta}.
\]

In order to compute \( q_i^* \), we suppose that it has the form of

\[
q_1^* = N_1 \left( \frac{d}{1} \right) e^{i \omega_1 s}, q_2^* = N_2 \left( \frac{e}{1} \right) e^{i \omega_2 s}.
\]

By a direct calculation based on (3.3) and (3.5)–(3.8), we obtain

\[
q_1^* = \frac{1}{m_1 + il_1} \left( -b - i \omega_1 \right) e^{i \omega_1 s}, \quad q_2^* = \frac{1}{m_2 + il_2} \left( -b - i \omega_2 \right) e^{i \omega_2 s}
\]

with

\[
l_1 = -2\omega_1 + Ae^{-p \tau} \sin(\omega_1 \tau), \quad m_1 = -b + Ae^{-p \tau} \cos(\omega_1 \tau),
\]

\[
l_2 = -2\omega_2 + Ae^{-p \tau} \sin(\omega_2 \tau), \quad m_2 = -b + Ae^{-p \tau} \cos(\omega_2 \tau).
\]

Thus, the explicit expressions for \( \Phi \) and \( \Psi \) can be obtained by substituting \( q_1, q_i^* \) \((i = 1, 2)\) into Eq. (3.6).

Let \( z = (z_1, \bar{z}_1, z_2, \bar{z}_2)^T \) and \( v_i \in Q_A \cap C([-\tau, 0), R) \), and denote \( u_i = \Phi z + v_i \). Equation (3.4) is equivalent to the following system

\[
\begin{aligned}
z' &= Bz + \Psi(0)(A(p, \tau)(\Phi z + v_i) + \varepsilon F(\Phi z + v_i)), \\
v_i' &= A(0)v_i + (I - \pi)X_0(A(p, \tau)(\Phi z + v_i) + \varepsilon F(\Phi z + v_i)),
\end{aligned}
\]

where

\[
B = \begin{bmatrix}
i \omega_1 & 0 & 0 & 0 \\
0 & -i \omega_1 & 0 & 0 \\
0 & 0 & i \omega_2 & 0 \\
0 & 0 & 0 & -i \omega_2
\end{bmatrix}.
\]
The matrix $B$ generates the torus group $T^2 = S^1 \times S^1$ whose action on $C^2$ is given by

$$(\theta_1, \theta_2)(z_1, z_2) = (e^{i\theta_1}z_1, e^{i\theta_2}z_2).$$

Then the $T^2$-equivalent normal form, which is truncated to the quadratic order, is

$$\begin{cases}
z'_1 = i\omega_1 z_1 + b_1^1 p_\varepsilon z_1 + b_2^1 \tau_\varepsilon z_1, \\
z'_2 = i\omega_2 z_2 + b_1^2 p_\varepsilon z_2 + b_2^2 \tau_\varepsilon z_2,
\end{cases} \quad (3.10)$$

with

$$\begin{align*}
b_1^1 &= \frac{1}{m_1 - il_1} e A e^{-p_\varepsilon \tau_c} i \tau_c (\sin(\omega_1 \tau_c) + i \cos(\omega_1 \tau_c)), \\
b_1^2 &= \frac{1}{m_1 - il_1} e A e^{-p_\varepsilon \tau_c} (-\omega_1 + bi)(\sin(\omega_1 \tau_c) + i \cos(\omega_1 \tau_c)), \\
b_2^1 &= \frac{1}{m_2 - il_2} e A e^{-p_\varepsilon \tau_c} i \tau_c (\sin(\omega_2 \tau_c) + i \cos(\omega_2 \tau_c)), \\
b_2^2 &= \frac{1}{m_2 - il_2} e A e^{-p_\varepsilon \tau_c} (-\omega_2 + bi)(\sin(\omega_2 \tau_c) + i \cos(\omega_2 \tau_c)).
\end{align*}$$

In polar coordinates $z_1 = r_1 e^{i\rho_1}$ and $z_2 = r_2 e^{i\rho_2}$, the amplitude equation resulted from Eq. (3.10) is

$$\begin{align*}
r'_1 &= (\Re(b_1^1) p_\varepsilon + \Re(b_2^1) \tau_\varepsilon) r_1, \\
r'_2 &= (\Re(b_1^2) p_\varepsilon + \Re(b_2^2) \tau_\varepsilon) r_2.
\end{align*}$$

If removing the dependence on the unfolding parameters $p_\varepsilon$, $\tau_\varepsilon$ from Eq. (3.9), we obtain

$$\begin{cases}
z'_1 = i \omega_1 z_1 + \tilde{\psi}_1^T(0) F(\Phi z + v_1), \\
z'_2 = i \omega_2 z_2 + \tilde{\psi}_2^T(0) F(\Phi z + v_1), \\
v'_1 = A(0) v_1 + (I - \pi) X_0(\varepsilon F(\Phi z + v_1)).
\end{cases} \quad (3.11)$$

In order to determine the phase portraits of system (3.11) around the equilibria, we need to use higher order terms in the Taylor series as well as the center manifold theorem. For system (3.11) the third order terms are sufficient and the truncated power series of $v_1$ on the center manifold can be approximated by zero. Hence the resulting truncated system of (3.11) becomes

$$\begin{cases}
z'_1 = i \omega_1 z_1 + \tilde{\psi}_1^T(0) F(\Phi z) \\
&= i \omega_1 z_1 + c_1(z_1 + \bar{z}_1 + z_2 + \bar{z}_2)^2[i \gamma \omega_1 (z_1 - \bar{z}_1) + i \gamma \omega_2 (z_2 - \bar{z}_2) + \beta(z_1 + \bar{z}_1 + z_2 + \bar{z}_2)], \\
z'_2 = i \omega_2 z_2 + \tilde{\psi}_2^T(0) F(\Phi z) \\
&= i \omega_2 z_2 + c_2(z_1 + \bar{z}_1 + z_2 + \bar{z}_2)^2[i \gamma \omega_1 (z_1 - \bar{z}_1) + i \gamma \omega_2 (z_2 - \bar{z}_2) + \beta(z_1 + \bar{z}_1 + z_2 + \bar{z}_2)],
\end{cases} \quad (3.12)$$

with

$$c_1 = -\varepsilon \frac{1}{m_1 - il_1}, \quad c_2 = -\varepsilon \frac{1}{m_2 - il_2}. \quad (3.13)$$

Here we apply the standard normalization technique by setting

$$\begin{cases}
z_1 = \eta_1 + h_1(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2), \\
\eta'_1 = i \omega_1 \eta_1 + B_{11} \eta_1 \eta_2 + B_{12} \eta_1 \bar{\eta}_2,
\end{cases} \quad \begin{cases}
z_2 = \eta_2 + h_2(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2), \\
\eta'_2 = i \omega_2 \eta_2 + B_{21} \eta_1 \bar{\eta}_2 + B_{22} \eta_1 \eta_2,
\end{cases} \quad (3.14)$$

where

$$h_1(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2) = \sum_{j+k+l+m=3} h_{jklm}^1 \eta_1^j \bar{\eta}_1^k \eta_2^l \bar{\eta}_2^m,$$

$$h_2(\eta_1, \bar{\eta}_1, \eta_2, \bar{\eta}_2) = \sum_{j+k+l+m=3} h_{jklm}^2 \eta_1^j \bar{\eta}_1^k \eta_2^l \bar{\eta}_2^m.$$
Substituting (3.13) and (3.14) into (3.12) and comparing the corresponding coefficients on both sides to determine $h_1$ and $h_2$ so as to round off the non-resonant terms, we obtain

$$
B_{11} = c_1(3\beta + i\gamma \omega_1), \quad B_{12} = 2c_1(3\beta + i\gamma \omega_1), \\
B_{21} = 2c_2(3\beta + i\gamma \omega_2), \quad B_{22} = c_2(3\beta + i\gamma \omega_2).
$$

Let

$$
\eta_1 = r_1e^{i\theta_1}, \quad \eta_2 = r_2e^{i\theta_2}.
$$

In polar coordinates, the amplitude equation resulted from Eq. (3.12) becomes

$$
\begin{align*}
\dot{r}_1' &= r_1(\text{Re}(B_{11})r_1^2 + \text{Re}(B_{12})r_2^2), \\
\dot{r}_2' &= r_2(\text{Re}(B_{21})r_1^2 + \text{Re}(B_{22})r_2^2).
\end{align*}
$$

Buono and Bélair [15] proved that the double Hopf bifurcation can be determined to the third order in the case where $\text{Re}(B_{11}) \neq 0$ and $\text{Re}(B_{22}) \neq 0$ as for the $\mathbb{Z}_2$-symmetric first-order scalar equation with two delays. By combining the results of Eq. (3.10) with Eq. (3.12), the normal form arising from Eq. (3.9) becomes

$$
\begin{align*}
\eta_1' &= i\omega_1\eta_1 + b_1^1p_\epsilon \eta_1 + b_1^2\tau_\epsilon \eta_1 + B_{11}\eta_1^2\eta_1 + B_{12}\eta_1\eta_2\eta_2, \\
\eta_2' &= i\omega_2\eta_2 + b_2^1p_\epsilon \eta_2 + b_2^2\tau_\epsilon \eta_2 + B_{21}\eta_1\eta_1\eta_2 + B_{22}\eta_2^2\eta_2,
\end{align*}
$$

where the coefficients of the resonant monomials in Eq. (3.10) are unchanged by the normal form transformation from Eq. (3.12). In polar coordinates $\eta_1 = r_1e^{i\theta_1}$, $\eta_2 = r_2e^{i\theta_2}$, the amplitude and phase equations can be derived from Eq. (3.15) as

$$
\begin{align*}
\dot{r}_1' &= r_1(\mu_1 + \text{Re}(B_{11})r_1^2 + \text{Re}(B_{12})r_2^2), \\
\dot{r}_2' &= r_2(\mu_2 + \text{Re}(B_{21})r_1^2 + \text{Re}(B_{22})r_2^2), \\
\dot{\theta}_1' &= \omega_1 + \upsilon_1 + \text{Im}(B_{11})r_1^2 + \text{Im}(B_{12})r_2^2, \\
\dot{\theta}_2' &= \omega_2 + \upsilon_2 + \text{Im}(B_{21})r_1^2 + \text{Im}(B_{22})r_2^2,
\end{align*}
$$

with $\mu_i = \text{Re}(b_i^1)p_\epsilon + \text{Re}(b_i^2)\tau_\epsilon$, $\upsilon_i = \text{Im}(b_i^1)p_\epsilon + \text{Im}(b_i^2)\tau_\epsilon$ for $i = 1, 2$.

Now we discuss the restriction on the phase portraits near the double Hopf bifurcation point. We rewrite Eq. (3.16) as

$$
\begin{align*}
\dot{r}_1' &= r_1(\mu_1 + \bar{a}r_1^2 + \bar{b}r_2^2), \\
\dot{r}_2' &= r_2(\mu_2 + \bar{c}r_1^2 + \bar{d}r_2^2),
\end{align*}
$$

where

$$
\begin{align*}
\bar{a} &= \frac{\text{Re}(B_{11})}{|\text{Re}(B_{11})|}, \quad \bar{b} = \frac{\text{Re}(B_{12})}{|\text{Re}(B_{12})|}, \quad \bar{c} = \frac{\text{Re}(B_{21})}{|\text{Re}(B_{21})|}, \quad \bar{d} = \frac{\text{Re}(B_{22})}{|\text{Re}(B_{22})|}.
\end{align*}
$$

After analyzing the pitchfork and Hopf bifurcations of Eq. (3.17), we classify eight regions and phase portraits in each region are illustrated in Fig. 3(A). The general classification of bifurcations can be obtained by analyzing secondary pitchfork bifurcations and Hopf bifurcations from nontrivial equilibria of (3.17). For example, we just choose $b = 0.27$, $\omega_0 = 0.8$, $\beta = 1.0$, $\gamma = 1.0$ and $\epsilon = 1.0$ in the following discussions.

(a) The $3 : 4$ weak resonant double Hopf bifurcation occurs at $p_\epsilon = 0.242056$, $\tau_\epsilon = 4.0691386$ with imaginary roots $i\omega_1 = 1.6395589i$, $i\omega_2 = 2.18608i$. Simultaneously, the unfolding parameters are $\mu_1 = -0.3397990370\tau_\epsilon - 0.8063129226p_\epsilon$ and $\mu_2 = 0.3997215768\tau_\epsilon - 0.5745888610p_\epsilon$. A straightforward computation on formulas (3.18) gives $\bar{a} = -1.0$, $\bar{d} = 1.0$, $\bar{b} = -4.018435652$ and $\bar{c} = 0.9954122312$.

Note that the trivial solution $(0, 0)$ of Eq. (3.17) is asymptotically stable in the dead regions. Pitchfork bifurcations occur at $(0, 0)$ along two lines $\mu_1 = 0$ and $\mu_2 = 0$, and pitchfork bifurcations also occur at $(\sqrt{\mu_1}, 0)$, along the line $\mu_2 + \bar{c}\mu_1 = 0$, and at $(0, \sqrt{-\mu_2})$ along the line $\mu_1 - \bar{b}\mu_2 = 0$. We can analyze the stability types of these bifurcations and draw the pitchfork bifurcation lines in the $(p, \tau)$ plane, as presented in Fig. 3(a). The shaded region $I$ in Fig. 3(a) is just “amplitude death island.” It can be verified that the periodic solution only exists in the unshaded regions.
Fig. 3. Classification and bifurcation sets for system (3.1) due to resonant and non-resonant double Hopf bifurcations in the \((p - \tau)\) plane. (a) The \(3:4\) double Hopf bifurcation point; (b) the \(4:5\) double Hopf bifurcation point; (c) the \(2:\sqrt{6}\) double Hopf bifurcation point; (d) the \(\sqrt{5}:3\) double Hopf bifurcation point. According to (a)–(d), the phase portraits on different regions are classified as shown in (A).

point solution \((r_1, r_2) = (\sqrt{-\frac{\mu_1 + b\mu_2}{cb+1}}, \sqrt{-\frac{\mu_2 + c\mu_1}{cb+1}})\) appears through pitchfork bifurcations, and the Hopf bifurcation may also occur at this fixed point. To detect such a behavior, the linearization at this fixed point is given by the matrix

\[
\begin{bmatrix}
\mu_1 - 3r_1^2 + br_2^2 & 2br_1r_2 \\
2cr_1r_2 & \mu_2 + cr_1^2 + 3r_2^2
\end{bmatrix}
\]
with the trace
\[
\frac{2((b - 1)\mu_2 - (c + 1)\mu_1)}{cb + 1},
\]
and the determinant
\[
\frac{-4(-\mu_1 + b\mu_2)(\mu_2 + c\mu_1)}{cb + 1}.
\]
Thus, we conclude that the Hopf bifurcation occurs on the line
\[
\mu_2 = \frac{(c + 1)\mu_1}{b - 1}.
\]
(b) The 4 : 5 weak resonant double Hopf bifurcation occurs at \(\tau_c = 5.474364, p_c = 0.161531\). After calculations we have \(i\omega_1 = 1.259855 i, i\omega_2 = 1.574819 i\). By virtue of a similar analysis, the unfolding parameters are \(\mu_1 = -0.258212726 - 0.832489 p_c, \mu_2 = -0.05111775 - 1.667528 p_c, \) and \(b = -2.935085, \bar{c} = 1.36282, \bar{a} = -1.0, \bar{d} = 1.0\). The pitchfork bifurcation lines \(\mu_1 = \mu_2 = 0, \mu_1 - \bar{b}\mu_2 = 0 \) and \(\mu_2 + \bar{c}\mu_1 = 0\) are plotted in the \((p, \tau)\) plane (see Fig. 3(b)), and the Hopf bifurcation occurs on the line \(\mu_2 = \frac{(c + 1)\mu_1}{b - 1}\).

(c) As shown in Fig. 3(c), the 2 : \(\sqrt{6}\) non-resonant double Hopf bifurcation occurs at \(\tau_c = 5.827051, p_c = 0.144807\), with \(i\omega_1 = 1.1960876 i\) and \(i\omega_2 = 1.464902 i\). The unfolding parameters are \(\mu_1 = -0.255348\tau_e - 0.845724 p_c, \mu_2 = 0.2955686\tau_e - 0.5159079 p_c, \) and \(b = -2.7680, \bar{c} = 1.445086, \bar{a} = -1.0, \bar{d} = 1.0\). The pitchfork bifurcation lines \(\mu_1 = \mu_2 = 0, \mu_1 - \bar{b}\mu_2 = 0 \) and \(\mu_2 + \bar{c}\mu_1 = 0\) are plotted in the \((p, \tau)\) plane (see Fig. 3(c)), and the Hopf bifurcation occurs on the line \(\mu_2 = \frac{(c + 1)\mu_1}{b - 1}\) as well.

(d) From Fig. 3(d), we can see that the \(\sqrt{5} : 3\) non-resonant double Hopf bifurcation occurs at \(\tau_c = 3.908103, p_c = 0.253733\). After calculations we have \(i\omega_1 = 1.701674 i, i\omega_2 = 2.283035 i\). The unfolding parameters are \(\mu_1 = -0.358030\tau_e - 0.804714 p_c, \mu_2 = 0.421327\tau_e - 0.577959 p_c, \) and \(b = -4.2293655, \bar{c} = 0.945768, \bar{a} = -1.0, \bar{d} = 1.0\). The pitchfork bifurcation lines \(\mu_1 = \mu_2 = 0, \mu_1 - \bar{b}\mu_2 = 0 \) and \(\mu_2 + \bar{c}\mu_1 = 0\) are illustrated in the \((p, \tau)\) plane, and the Hopf bifurcation occurs on the line \(\mu_2 = \frac{(c + 1)\mu_1}{b - 1}\) as well.

In Fig. 3, parts (a)–(d) provide us useful information on classification of dynamical behaviors near the double Hopf point \((p_c, \tau_c)\). Numerical simulations of complex dynamical behaviors from regime I to VI are shown in part (A). It is seen that quasi-periodic solutions of Eq. (3.1) exist in region V as shown in Fig. 4. By choosing the Poincaré section given by \(\dot{x}(t - \tau) = 0\), the closed curves on the Poincaré section verify the existence of the corresponding quasi-periodic attractors. The Neimark–Sacker bifurcation occurs on the curve HF as shown in Fig. 3. We consider the 3 : 4 double Hopf bifurcation points shown in Fig. 3(a) firstly, then fix \(\tau = 4.0\) and let the parameter \(p\) vary and

![Fig. 4. The quasi-periodic solutions near double Hopf bifurcation points. (a) For parameters \(p = 0.16, \tau = 5.36\) in region V near the 4 : 5 double Hopf points; (b) for parameters \(p = 0.146, \tau = 5.76\) in the region V near the 2 : \(\sqrt{6}\) double Hopf points.](image-url)
cross $HF$ from region $VI$ to region $V$. It is shown that a chaotic attractor appears through breaking-down of the quasi-periodic attractor. By choosing the Poincaré section given by $\dot{x}(t - \tau) = 0$ in the phase space, a Neimark–Sacker bifurcation is observed as shown in Fig. 5. For the $\sqrt{5}:3$ double Hopf bifurcation points as shown in Fig. 3(d), we assume $\tau = 3.8$ and choose the Poincaré section $x(t - \tau) = 0$. A chaotic attractor appears through breaking-down of the quasi-periodic attractor via a Neimark–Sacker bifurcation in Fig. 6.
Fig. 6. Phase portraits near the $\sqrt{5}:3$ double Hopf bifurcations. (a) Quasi-periodic attractor; (b) transient state; (c) chaotic solutions. A Neimark–Sacker bifurcation is experienced on the Poincaré section as shown in figures (a1)–(b1)–(c1).

4. Conclusion

In this work, a van der Pol-Duffing oscillator subjected to the parametric delay feedback control is analyzed. The occurrence of delay dependent parameters would greatly complicate the analysis of stability and bifurcation of delay systems. A geometrical method as well as an analytical method is presented to determine the stability condition and stability switches of the equilibrium state, and the double Hopf bifurcations are also discussed. The main results are listed as follows:
Firstly, the Hopf bifurcation is strongly relied on the parameter $p$, except for the time delay $\tau$. The transverse direction of variation of the imaginary characteristic roots are determined by the variation of the parameter $p$ with respect to the variable $\theta$.

Secondly, the Hopf bifurcation curves are illustrated in the $(\tau, p)$ plane, and the multi-stability regions are obtained. Some weak resonant and non-resonant double Hopf bifurcation phenomena arise due to two pairs of imaginary characteristic roots appearing on the margin of the “death islands” simultaneously.

Finally, the dynamics near the double Hopf bifurcation points are classified and verified by numerical simulations. It is exhibited that chaos can be induced by quasi-periodic solutions due to the Neimark–Sacker bifurcation on the Poincaré section.

References