Local Hopf bifurcation and global existence of periodic solutions in a kind of physiological system

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Received 6 May 2006; accepted 28 July 2006

Abstract

The dynamics of a physiological control systems described by a first-order nonlinear delay differential equations are investigated. we proved that a sequence of Hopf bifurcations occur at the positive equilibrium as the delay increases. Explicit algorithm for determining the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions are derived, using the theory of normal form and center manifold. Global existence of periodic solutions are established using a global Hopf bifurcation result due to Wu [Symmetric functional differential equations and neural networks with memory, Trans. Amer. Math. Soc. 350 (1998) 4799–4838].

MSC: 34K18; 34C20; 34K13; 34C25; 34C60

Keywords: Physiological control system; Global periodic solution; Hopf bifurcation; Stability

1. Introduction

The equation

$$\dot{x}(t) = \gamma - \frac{\beta v_m x(t) x^n(t - \tau)}{\theta^2 + x^n(t - \tau)}, \quad t \geq 0$$

(1.1)

was used by Mackey and Glass [7,2] to describe a physiological control system. Here $x(t)$ denotes the arterial CO2 concentration, $\tau$ is the time delay between oxygenation of blood in the lungs and stimulation of chemoreceptors in the brainstem, $\gamma$ is the CO2 production rate, $\beta$, $\theta$ and $v_m$ are all positive constants. The equation reproduces certain qualitative features of normal and abnormal respiration. In the paper, Gopalsamy et al. [3] obtained sufficient conditions for all positive solutions of system (1.1) to oscillate about the positive equilibrium of system (1.1) and sufficient conditions of global attractively.

The main purpose of the paper is to show that simple mathematical models of physiological systems predict the existence or regimes of periodic and aperiodic dynamics, similar to those encountered in human disease. It is known that periodic solutions can arise though the Hopf bifurcation in delay differential equation [5,16,4]. However, these
periodic solutions bifurcating are generally local. Therefore, it is an important mathematical subject to investigate if these nonconstant periodic solution which are obtained through local Hopf bifurcations exist globally.

A question of mathematical and biological interest is whether stable and sustained oscillations are possible for Eq. (1.1). In the present paper, we provide a detailed analysis of this question. Using the delay $\tau$ as a parameter, and applying the local and global Hopf bifurcation theory (see e.g. [5,16]), we investigate the existence of stable periodic oscillations for Eq. (1.1). More specifically, we prove that, as the delay $\tau$ increases, the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur. Furthermore, using the normal form and center manifold theory (see e.g. [6]), we derive an explicit algorithm and sufficient conditions for the stability of the bifurcating periodic solutions. Existence of periodic solutions for $\tau$ far away from the Hopf bifurcation values is also established, using a global Hopf bifurcation result of Wu [16].

We would like to mention that there are several articles on the global existence of periodic solutions in delayed differential equations based on the global Hopf bifurcation theory due to Wu [16], see [8,10–15].

The remainder of the paper is organized as follows: in Section 2, we investigate the stability of the positive equilibrium and the occurrence of Hopf bifurcations. In Section 3, direction and stability of the Hopf bifurcation are established. Global Hopf bifurcation is established in Section 4. In Section 5, we shall give some numerical simulations.

2. Stability of the positive equilibrium and local Hopf bifurcations

In this section, we shall study the stability of the positive equilibrium and the existence of local Hopf bifurcations. Under transformation $x(t) = \theta y(t)$, Eq. (1.1) becomes

$$
\dot{y}(t) = a - \frac{by(t)y^n(t-\tau)}{1+y^n(t-\tau)},
$$

(2.1)

where $a = r/\theta$ and $b = \beta \nu_m$. One can see that if $y_*$ is an equilibrium to Eq. (2.1), then $y_*$ satisfies

$$
by_*^{n+1} - ay_*^n - a = 0.
$$

Suppose $b > 0$. Set $F(y) = by_*^{n+1} - ay_*^n - a$. Then $F'(y) = y_*^{n-1}(b(n+1)y - an)$. So $F'(y) < 0$, $y \in (0, an/b(n+1))$, $F'(y) > 0$, $y \in (an/b(n+1), \infty)$. This and $F(0) = -a < 0$ and $\lim_{y \to \infty} F(y) = \infty$ imply that Eq. (2.1) has an unique positive equilibrium, denoted by $y_*$. Set $v(t) = y(t) - y_*$. Then $v(t)$ satisfies

$$
\dot{v}(t) = a - \frac{b(v(t) + y_*)(v(t-\tau) + y_*)^n}{1 + (v(t-\tau) + y_*)^n}.
$$

(2.2)

The linearization of Eq. (2.2) at $v = 0$ is

$$
\dot{v}(t) = -\frac{by_*^n}{1+y_*^n}v(t) - \frac{bny_*^n}{(1+y_*^n)^2}v(t-\tau)
$$

(2.3)

whose characteristic equation is

$$
\lambda = -\frac{by_*^n}{1+y_*^n} - \frac{bny_*^n}{(1+y_*^n)^2}e^{-\lambda \tau}.
$$

(2.4)

For $\tau = 0$, the only root of (2.4) is $\lambda = -(n+1)by_*^n/(1+y_*^n)^2 < 0$, since $n$, $b$ and $y_*$ are all positive constants. For $\omega > 0$, $i\omega$ is a root of (2.4) if and only if

$$
i\omega = -\frac{by_*^n}{1+y_*^n} - \frac{bny_*^n}{(1+y_*^n)^2}e^{-i\omega \tau}.
$$

Separating the real and imaginary parts, we get

$$
\begin{align*}
-\frac{by_*^n}{1+y_*^n} &= \frac{bny_*^n}{(1+y_*^n)^2} \cos \omega \tau, \\
\omega &= \frac{bny_*^n}{(1+y_*^n)^2} \sin \omega \tau.
\end{align*}
$$

(2.5)
which leads to
\[
\left( \frac{by_a^n}{1+y_a^n} \right)^2 + \omega^2 = \frac{b^2n^2y_a^{2n}}{(1+y_a^n)^2},
\]
namely,
\[
\omega = \frac{by_a^n}{(1+y_a^n)^2}\sqrt{n^2 - (1+y_a^n)^2}.
\]
This is possible if and only if \(n > 1 + y_a^n\).
For \(n > 1 + y_a^n\), let
\[
\tau_k = \frac{(1+y_a^n)^2}{by_a^n\sqrt{n^2 - (1+y_a^n)^2}} \cos^{-1}\left( -\frac{(1+y_a^n)}{n} + 2k\pi \right), \quad k = 0, 1, \ldots.
\]
Set
\[
\omega_0 = \frac{by_a^n}{(1+y_a^n)^2}\sqrt{n^2 - (1+y_a^n)^2}.
\]
Let \(\lambda_k(\tau) = \zeta_k(\tau) + i\omega_k(\tau)\) be a root of (2.4) near \(\tau = \tau_k\) satisfying \(\zeta_k(\tau_k) = 0\) and \(\omega_k(\tau_k) = \omega_0\). We have the following result.

**Lemma 2.1.** \(\dot{\zeta}_k(\tau_k) > 0\).

**Proof.** Suppose \(\zeta = -by_a^n/(1+y_a^n)\) and \(v = -bny_a^n/(1+y_a^n)^2\). Differentiating both sides of (2.4) with respect to \(\tau\), it follows that
\[
\frac{d\dot{\lambda}}{d\tau} = -ve^{-2\tau} \left( \dot{\lambda} + \tau \frac{d\dot{\lambda}}{d\tau} \right).
\]
Therefore, note that \(ve^{-2\tau} = \lambda - \zeta\), we have
\[
\frac{d\dot{\lambda}}{d\tau} = -ve^{-2\tau} \frac{d\lambda}{1 + vr e^{-2\tau}} = -\dot{\lambda}(\lambda - \zeta)
\]
and hence
\[
\frac{d\dot{\lambda}}{d\tau} \bigg|_{\tau=\tau_k} = \frac{\omega_0^2 + i\omega_0\lambda_k}{1 - \zeta\tau_k + i\omega_0\tau_k}.
\]
This implies that
\[
\dot{\zeta}(\tau_k) = \frac{\omega_0^2}{A},
\]
where \(A = (1 - \zeta\tau_k)^2 + (\omega_0\tau_k)^2\), completing the proof. 

**Lemma 2.2.** (i) If \(n < 1 + y_a^n\), then all roots of the characteristic Eq. (2.4) have negative real parts. (ii) If \(n > 1 + y_a^n\), then Eq. (2.4) has a pair of simple imaginary roots \(\pm i\omega_0\) when \(\tau = \tau_k, k = 0, 1, 2, \ldots\). Furthermore, if \(\tau \in [0, \tau_0)\), then all the roots of Eq. (2.4) have negative real parts; if \(\tau = \tau_0\), then all roots of (2.4) except \(\pm i\omega_0\) have negative real parts; and if \(\tau \in (\tau_k, \tau_{k+1})\) for \(k = 0, 1, 2, \ldots\), Eq. (2.4) has \(2(k+1)\) roots with positive real parts.

**Proof.** From the analysis leading to relation (2.6) we know that, if \(n < 1 + y_a^n\), Eq. (2.4) has no purely imaginary root \(i\omega\) with \(\omega > 0\). Since \(\lambda = 0\) is not a root of (2.4), we know that, for any \(\tau \geq 0\), Eq. (2.4) has no roots on the imaginary axis. Applying a result of Ruan and Wei [9, Corollary 2.4], we arrive at the conclusion (i).
If \( n > 1 + y_*^n \), let \( \tau_k \) be as in (2.7). From (2.5) and (2.6), we have that Eq. (2.4) has purely imaginary roots \( \pm i\omega_0 \) if and only if \( \tau = \tau_k \) and \( \omega_0 \) is given in (2.8). The statement on the number of eigenvalues with positive real parts follow from Lemma 2.1 and Rouche’s Theorem [1, Theorem 9.17.4]. □

Spectral properties in Lemma 2.2 immediately lead to stability properties of the zero solution of Eq. (2.2), and equivalently, of the positive equilibrium \( y = y_* \) of Eq. (2.1).

**Theorem 2.1.** For Eq. (2.1), the followings hold:

(i) If \( n < 1 + y_*^n \), then \( y = y_* \) is asymptotically stable for any \( \tau > 0 \).
(ii) If \( n > 1 + y_*^n \), then \( y = y_* \) is asymptotically stable for \( \tau \in [0, \tau_0) \), and unstable for \( \tau > \tau_0 \).
(iii) For \( n > 1 + y_*^n \), Eq. (2.1) undergoes a Hopf bifurcation at \( y_* \) when \( \tau = \tau_k \), for \( k = 0, 1, 2, \ldots \).

### 3. Stability and direction of the Hopf bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when \( \tau = \tau_k, k = 0, 1, 2, \ldots \). In this section we shall derive the explicit formulae determining the direction, stability, and period of these periodic solutions bifurcating from equilibrium \( y_* \) at these critical values of \( \tau \), by using techniques from normal form and center manifold theory [6]. Without loss of generality, denote any one of these critical values \( \tau = \tau_k \), \( k = 0, 1, 2, \ldots \) by \( \bar{\tau} \), at which Eq. (2.1) undergoes a Hopf bifurcation from \( y_* \).

Let \( u(t) = v(\bar{\tau}t) \). Then Eq. (2.2) becomes

\[
\dot{u}(t) = a\tau - \frac{b\tau(u(t) + y_*)(u(t - 1) + y_*)^n}{1 + (u(t - 1) + y_*)^n}.
\]  

(3.1)

Correspondingly, the characteristic (2.4) becomes

\[
z = -\frac{b\tau y_*^n}{1 + y_*^n} - \frac{b\tau ny_*^n}{(1 + y_*^n)^2} e^{-z},
\]  

(3.2)

with \( z = \lambda \tau \) for \( \tau \neq 0 \). By Lemma 2.1 we know that, if \( n > 1 + y_*^n \) the root of Eq. (3.2)

\[
z(\tau) = \tau \lambda(\tau) + i\lambda_0(\tau)
\]

with \( \lambda(\bar{\tau}) = 0 \) and \( \lambda_0(\bar{\tau}) = \lambda_0 \) satisfies

\[
\frac{d\tau \lambda(\tau)}{d\tau} \bigg|_{\tau = \bar{\tau}} = \bar{\tau} \lambda'(\bar{\tau}) > 0.
\]

Set \( \tau = \bar{\tau} + \mu, \mu \in R \). Then \( \mu = 0 \) is a Hopf bifurcation value for Eq. (3.1). Rewrite Eq. (3.1) as

\[
\dot{u}(t) = C_1(\mu)u(t) + C_2(\mu)u(t - 1) + C_3(\mu)u^2(t - 1) + C_4(\mu)u^3(t - 1) + C_5(\mu)u(t)u^2(t - 1) + C_6(\mu)u(t)u^2(t - 1) + O(u^4).
\]  

(3.3)

Here

\[
C_1(\mu) = -\frac{b(\bar{\tau} + \mu)y_*^n}{1 + y_*^n},
\]

\[
C_2(\mu) = -\frac{b(\bar{\tau} + \mu)ny_*^n}{(1 + y_*^n)^2},
\]

\[
C_3(\mu) = -b(\bar{\tau} + \mu) \left[ n(n - 1)\frac{y_*^{n-1}}{2(1 + y_*^n)} - n^2\frac{y_*^{2n-1}}{(1 + y_*^n)^2} + n(n - 1)\frac{y_*^{2n-1}}{2(1 + y_*^n)^2} \right],
\]

(3.4)
\[ C_4(\mu) = b(\overline{\mu}) + \mu \left\{ \left[ (n - 1)(n - 2) - 4(n - 1)(n + 1)y_n^2 + (n + 2)(n + 2)y_n^2 - 6(1 + y_n^2)^3 \right] \frac{ny_{n-1}^2}{6(1 + y_n^2)^3} \right. \]
\[ + n^2(n - 1) \left( 1 - y_n^2 \right) - n^2(n - 1) \left( 1 - y_n^2 \right) - n(n - 1)(n - 2) \frac{y_{n-2}^2}{6(1 + y_n^2)^2} \right\}, \]
\[ C_5(\mu) = -b(\overline{\mu})ny_n^{n-1}, \]
\[ C_6(\mu) = \frac{2b(\overline{\mu})(n^2y_n^{n-2} - n(n - 1)y_n^{n-2}(1 + y_n^2))}{(1 + y_n^2)^3}. \]

For \( \varphi \in C([-1, 0], R) \), let
\[ L_\mu \varphi = C_1(\mu)\varphi(0) + C_2(\mu)\varphi(-1) \]
and
\[ f(\mu, \varphi) = C_3(\mu)u^2(-1) + C_4(\mu)u^3(-1) + C_5(\mu)u(0)u(-1) + C_6(\mu)u(0)u^2(-1) + O(u^4). \]

By the Riesz Representation Theorem, there exists a function \( \eta(\theta, \mu) \) of bounded variation for \( \theta \in [-1, 0] \) such that
\[ L_\mu \varphi = \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta) \quad \text{for} \quad \varphi \in C. \]

In fact, we can choose
\[ \eta(\theta, \mu) = C_1(\mu)\delta(\theta) + C_2(\mu)\delta(\theta + 1). \]

For \( \varphi \in C^1([-1, 0], R) \), we set
\[ A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\xi, \mu)\varphi(\xi), & \theta = 0 \end{cases} \]

and
\[ N(\mu)\varphi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \varphi), & \theta = 0. \end{cases} \]

Then (3.1) can be rewritten as
\[ \dot{u}_t = A(\mu)u_t + N(\mu)u_t, \]
where \( u_t(\theta) = u(t + \theta) \) for \( \theta \in [-1, 0] \). For \( \psi \in C^1([0, 1], R) \), define
\[ A^*\psi = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta(t, 0)\psi(-t), & s = 0. \end{cases} \]

For \( \varphi \in C[-1, 0] \) and \( \psi \in C[0, 1] \), using the bilinear form
\[ (\psi, \varphi) = \tilde{\psi}(0)\varphi(0) - \int_{-1}^{0} \int_{\xi=0}^{s} \tilde{\psi}(\xi - \theta) d\eta(\theta)\varphi(\xi) d\xi \]
we know that \( A^* \) and \( A = A(0) \) are adjoint operators. By the discussion at the beginning of this section, we know that \( \pm i\tau_{\omega_0} \) are eigenvalues of \( A(0) \). Thus they are eigenvalues of \( A^* \). Clearly, \( q(\theta) = e^{i\tau_{\omega_0}\theta} \) is an eigenvector of \( A \) corresponding to the eigenvalue \( i\tau_{\omega_0} \), and \( q^*(s) = De^{i\tau_{\omega_0}s} \) is an eigenvector of \( A^* \) corresponding to the eigenvalue
\( D = \frac{1}{1 + b\bar{\gamma}q^2/(1 + \gamma^2)} - i\bar{\tau}w_0. \) Furthermore,
\[
\langle q^*, q \rangle = 1, \quad \langle q^*, \bar{q} \rangle = 0,
\]
where
\[
W(t) = u_t + 2\text{Re}\{z(t)q(\theta)\}.
\]

On the center manifold \( C_0 \) we have
\[
W(t, \theta) = W(z(t), \bar{z}(t), \theta),
\]
where
\[
W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots,
\]
where \( z \) and \( \bar{z} \) are local coordinates for center manifold \( C_0 \) in the direction of \( q^* \) and \( \bar{q}^* \). Note that \( W \) is real if \( u_t \) is real. We consider only real solutions. For solutions \( u_t \in C_0 \) of (3.1), since \( \mu = 0 \),
\[
\dot{z}(t) = i\omega_0 z + \bar{q}^*(\theta)f(0, w(z, \bar{z}, \theta) + 2\text{Re}\{zq(\theta)\})
\]
that is
\[
\dot{z}(t) = i\omega_0 z + g(z, \bar{z}),
\]
where
\[
g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots,
\]
\[
g(z, \bar{z}) = \bar{q}^*(0)f_0(z, \bar{z}) + f(0, u_t)
\]
\[
= C_3 \bar{D} \left( zq(-1) + \bar{z}\bar{q}(1) + W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} \right)^2
\]
\[
+ C_4 \bar{D} \left( zq(-1) + \bar{z}\bar{q}(1) + W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} \right)^3
\]
\[
+ C_5 \bar{D} \left( zq(0) + \bar{z}\bar{q}(0) + W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} \right)
\]
\[
\times \left( zq(-1) + \bar{z}\bar{q}(1) + W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} \right)
\]
\[
+ C_6 \bar{D} \left( zq(0) + \bar{z}\bar{q}(0) + W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} \right)
\]
\[
\times \left( zq(-1) + \bar{z}\bar{q}(1) + W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} \right)^2
\]
\[
= \bar{D} \left[ C_3q^2(-1) + C_5q(0)q(-1) \right] z^2 + \bar{D} \left[ 2C_3q(-1)\bar{q}(1) + C_5(q(0)\bar{q}(1) + \bar{q}(0)q(-1)) \right] z\bar{z}
\]
\[
+ \bar{D} \left[ C_3\bar{q}^2(-1) + C_5\bar{q}(0)\bar{q}(1) \right] \bar{z}^2 + \bar{D} \left[ C_3[\bar{q}(1)W_{20}(\theta) + 2q(-1)W_{11}(\theta)]
\]
\[
+ 3C_4q^2(-1)\bar{q}(1) + C_5 \left[ q(0)W_{11}(\theta) + \frac{1}{2}\bar{q}(1)W_{20}(\theta) + \frac{1}{2}\bar{q}(1)W_{02}(\theta) + q(-1)W_{11}(\theta) \right]
\]
\[
+ C_6 \left[ q^2(-1)\bar{q}(0) + 2q(0)q(-1)\bar{q}(1) \right] \right] z^2 + \cdots
So we obtain

\[ g_{20} = 2\tilde{D} \left( C_3 e^{-2i\tau_0} + C_5 e^{-i\tau_0} \right), \]
\[ g_{11} = \tilde{D} \left[ 2C_3 + C_5 \left( e^{i\tau_0} + e^{-i\tau_0} \right) \right], \]
\[ g_{02} = 2\tilde{D} \left( C_3 e^{2i\tau_0} + C_5 e^{i\tau_0} \right), \]
\[ g_{21} = 2\tilde{D} \left\{ C_3 \left[ e^{i\tau_0} W_{20}(-1) + 2e^{-i\tau_0} W_{11}(-1) \right] + 3C_4 e^{-i\tau_0} \right. \]
\[ \left. + C_5 \left[ W_{11}(-1) + \frac{1}{2} W_{20}(-1) + \frac{1}{2} e^{i\tau_0} W_{20}(0) + e^{-i\tau_0} W_{11}(0) \right] \right\} + C_6 \left( e^{-2i\tau_0} + 2 \right). \]  

(3.14)

Since there are \( W_{20}(0) \) and \( W_{11}(0) \) in \( g_{21} \), we still need to compute them.

From (3.9) and (3.12), we have

\[ \dot{W} = \dot{u}(t) - \dot{z}q - \dot{z}\tilde{q} = \begin{cases} \{ AW - 2 \text{Re}\{ \tilde{q}^* (0) f_0 q(\theta) \} \}, & \theta \in [-1, 0), \\ \{ AW - 2 \text{Re}\{ \tilde{q}^* (0) f_0 q(0) \} + f_0 \}, & \theta = 0 \end{cases} \]

\[ =: AW + H(Z, \bar{Z}, \theta), \]

where

\[ H(z(t), \bar{z}(t), \theta) = H_{20}(\theta) \frac{z \bar{z}}{2} + H_{11}(\theta) z \bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots \]  

(3.15)

comparing the coefficients, we obtain

\[ (A - 2i\omega_0 \tilde{\tau}) W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta) \]  

(3.16)

and we know that for \( \theta \in [-1, 0) \),

\[ H(z(t), \bar{z}(t), \theta) = -\tilde{q}^* (0) f_0 q(\theta) - q^* (0) \tilde{f}_0 \tilde{q}(\theta) = -g(z, \bar{z}) q(\theta) - \bar{g}(z, \bar{z}) \tilde{q}(\theta). \]  

(3.17)

Comparing the coefficients with (3.15) gives that

\[ H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02} \tilde{q}(\theta) \]  

(3.18)

and

\[ H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11} \tilde{q}(\theta). \]  

(3.19)

From (3.16) and (3.18), we get

\[ \dot{W}_{20}(\theta) = 2i\omega_0 \tilde{\tau} W_{20}(\theta) + g_{20} q(\theta) + \bar{g}_{02} \tilde{q}(\theta). \]

Note that \( q(\theta) = e^{i\omega_0 \tilde{\tau} \theta} \), hence

\[ W_{20}(\theta) = \frac{i g_{20}}{\omega_0 \tilde{\tau}} e^{i\omega_0 \tilde{\tau} \theta} + \frac{i g_{02}}{3 \omega_0 \tilde{\tau}} e^{-i\omega_0 \tilde{\tau} \theta} + E_1 e^{i\omega_0 \tilde{\tau} \theta}. \]  

(3.20)

Similarly, from (3.16) and (3.19), we have

\[ \dot{W}_{11}(\theta) = g_{11} q(\theta) + \bar{g}_{11} \tilde{q}(\theta) \]
and
\[ W_{11}(\theta) = -\frac{ig_{11}}{\omega_0 \tau} e^{i\omega_0 \tau \theta} + \frac{ig_{11}}{\omega_0 \tau} e^{-i\omega_0 \tau \theta} + E_2. \] (3.21)

In what follows we shall seek appropriate \( E_1 \) and \( E_2 \) in (3.20) and (3.21), respectively. It follows from the definition of \( A \) and (3.18) that
\[ \int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\omega_0 \tau W_{20}(0) - H_{20}(0), \] (3.22)
\[ \int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \] (3.23)
where \( \eta(\theta) = \eta(0, \theta) \).

From (3.15), we have
\[ H_{20}(0) = -g_{20} \dot{q}(0) - \ddot{g}_{02} \dot{q}(0) + 2C_3(0)e^{-2i\omega_0 \tau} + 2C_5(0)e^{-i\tau_0 \omega_0} \] (3.24)
and
\[ H_{11}(0) = -g_{11} q(0) - \dot{g}_{11} \dot{q}(0) + 2C_3(0) + C_5(0) \left(e^{i\tau_0 \omega_0} + e^{-i\tau_0 \omega_0}\right). \] (3.25)
Substituting (3.20) and (3.24) into (3.22), we obtain
\[ \left(C_1(0) + C_2(0)e^{-2i\omega_0 \tau} - 2i\omega_0 \tau\right) E_1 = 2C_3(0)e^{-2i\omega_0 \tau} + 2C_5(0)e^{-i\tau_0 \omega_0} \]
and
\[ \left(C_1(0) + C_2(0)e^{-2i\omega_0 \tau}\right) E_2 = -2C_3(0) + C_5(0) \left(e^{i\tau_0 \omega_0} + e^{-i\tau_0 \omega_0}\right) \]
so
\[ W_{20}(\theta) = \frac{ig_{20}}{\omega_0 \tau} e^{i\omega_0 \tau \theta} + \frac{ig_{02}}{3\omega_0 \tau} e^{-i\omega_0 \tau \theta} + \frac{2C_3(0) + 2C_5(0)e^{i\tau_0 \omega_0}}{C_1(0) + C_2(0)e^{-2i\omega_0 \tau} - 2i\omega_0 \tau}, \] (3.26)
\[ W_{11}(\theta) = -\frac{ig_{11}}{\omega_0 \tau} e^{i\omega_0 \tau \theta} + \frac{ig_{11}}{\omega_0 \tau} e^{-i\omega_0 \tau \theta} - \frac{2C_3(0) + C_5(0) \left(e^{i\tau_0 \omega_0} + e^{-i\tau_0 \omega_0}\right)}{C_1(0) + C_2(0)e^{-2i\omega_0 \tau}}. \] (3.27)
Substituting \( \theta = -1 \) into (3.26) and (3.27), respectively, we obtain
\[ W_{20}(-1) = \frac{ig_{20}}{\omega_0 \tau} e^{-i\omega_0 \tau} + \frac{ig_{02}}{3\omega_0 \tau} e^{i\omega_0 \tau} + \frac{2C_3(0) + 2C_5(0)e^{i\tau_0 \omega_0}}{C_1(0) + C_2(0)e^{-2i\omega_0 \tau} - 2i\omega_0 \tau}, \] (3.28)
\[ W_{11}(-1) = -\frac{ig_{11}}{\omega_0 \tau} e^{-i\omega_0 \tau} + \frac{ig_{11}}{\omega_0 \tau} e^{i\omega_0 \tau} - \frac{2C_3(0) + C_5(0) \left(e^{i\tau_0 \omega_0} + e^{-i\tau_0 \omega_0}\right)}{C_1(0) + C_2(0)e^{-2i\omega_0 \tau}}. \] (3.29)
Putting \( \theta = 0 \) into (3.26) and (3.27), respectively, we also obtain
\[ W_{20}(0) = \frac{ig_{20}}{\omega_0 \tau} + \frac{ig_{02}}{3\omega_0 \tau} + \frac{2C_3(0) + 2C_5(0)e^{i\tau_0 \omega_0}}{C_1(0) + C_2(0)e^{-2i\omega_0 \tau} - 2i\omega_0 \tau}, \] (3.30)
\[ W_{11}(0) = -\frac{ig_{11}}{\omega_0 \tau} + \frac{ig_{11}}{\omega_0 \tau} - \frac{2C_3(0) + C_5(0) \left(e^{i\tau_0 \omega_0} + e^{-i\tau_0 \omega_0}\right)}{C_1(0) + C_2(0)e^{-2i\omega_0 \tau}}. \] (3.31)
Theorem 3.1. The direction of the Hopf bifurcation of the system (2.1) at the equilibrium $x_0\text{ when } \tau = \tau_k \text{ (k = 0, 1, 2, ...)}$ is supercritical (subcritical) and the bifurcating periodic solutions on the center manifold are stable (unstable) if $Re(c'(\tilde{\tau})) > 0$ ($\beta_2 < 0$); particularly, when $\tau = \tau_0$, the stability of the bifurcating periodic solutions is the same as that on the center manifold.

4. Global existence of positive periodic solutions

In this section, we shall study the global continuation of positive periodic solutions bifurcating from the point $(y_*, \tau_k)$, $k = 0, 1, 2, \ldots$ for Eq. (2.1) with $n = 2k$, using global Hopf bifurcation theorem given by Wu [16]. For convenience of
the reader, we copy Eq. (2.1) in the following:
\[
\dot{y}(t) = a - \frac{by(t)y^n(t - \tau)}{1 + y^n(t - \tau)}.
\] (4.1)

At first, we introduce some notations:
\[
X = C([-\tau, 0], R),
\]
\[
\Sigma = Cl\{ (y, \tau, p) : u is a p - periodic solution of (40) \} \subset X \times R_+ \times R_+,
\]
\[
N = \{ (\hat{y}, \tau, p) : a = \frac{b\hat{y}^{n+1}}{1 + \hat{y}^n} \},
\]
\[
A = \lambda + \frac{by^n}{1 + y^n} + \frac{bny^n}{(1 + y^n)^2} e^{-\lambda \tau}.
\]

Let \( C(y_*, \tau_k, 2\pi/\omega_0) \) denote the connected component of \((y_*, \tau_k, 2\pi/\omega_0)\) in \( \Sigma \), where \( \tau_k \) and \( \omega_0 \) are defined in (2.7) and (2.8), respectively.

**Lemma 4.1.** All periodic solutions to (4.1) with \( n = 2k \) are positive and uniformly bounded.

**Proof.** Let \( y(t) \) be a nonconstant periodic solution to (4.1), and \( y(t_1) = M, y(t_2) = m \) be its maximum and minimum, respectively. Then \( y'(t_1) = y'(t_2) = 0 \), and by (4.1),
\[
m = a(1 + y^n(t_2 - \tau)) = a + \frac{a}{b} \frac{a}{b} > \frac{a}{b},
\]
\[
M = a(1 + y^n(t_1 - \tau)) = b + \frac{a}{b} \frac{a}{b} + \frac{a}{b} (a/b)^{n-1} < \frac{a}{b} + \frac{a}{b} (a/b)^{n-1}.
\]

Notice that \( n = 2k \) is an even number, hence \( y^n(t - \tau) \geq 0 \). It follows that \( a/b < y(t) < a/b + (b/a)^{n-1} \). This completes the proof. □

**Lemma 4.2.** Eq. (4.1) has no periodic solutions of period \( \tau \).

**Proof.** First, note that any nonconstant \( \tau \)-periodic solution \( u(t) \) of (4.1) is also a nonconstant periodic solution of the ordinary differential equation
\[
\dot{y}(t) = a - \frac{by(t)y^n(t)}{1 + y^n(t)}
\] (4.2)

when \( b > 0 \), Eq. (4.2) has unique equilibrium \( y^* \) and it is positive. It is well known that the positive equilibrium \( y^* \) is global asymptotically stable. Thus, there is not periodic solution. The above discussion means that (4.2) has no any nontrivial periodic solutions. It is a contradiction. Therefore, Lemma 4.2 is confirmed. □

**Theorem 4.1.** Suppose that \( n = 2j, j \geq 1 \) and \( b > 0 \) are satisfied. Then, for each \( \tau > \tau_k, k = 0, 1, 2, \ldots \), Eq. (4.1) has at least \( k \) periodic solutions, where \( \tau_k \) is defined in (2.7).

**Proof.** First, note that
\[
F(x^j, \tau, p) = a - \frac{by(t)y^n(t - \tau)}{1 + y^n(t - \tau)}
\]
satisfies the hypotheses \((A_1), (A_2) and (A_3)\) in Wu [16, p. 4813], with
\[
(\hat{y}_0, \omega_0, p_0) = \left(y_*, \tau_k, \frac{2\pi}{\omega_0}\right)
\]
Thus, when \( y_n \) is bounded, this contradiction completes the proof.

Clearly, if \(|\tau - \tau_k| \leq \delta\), then \( A_{(x_n, \tau, p)}(u + 2\pi / p) = 0\), then \( \tau = \tau_k\), \( u = 0\), and \( p = p_k\). This verifies the assumption \((A_4)\) in [16] for \( m = 1\). Moreover, putting

\[
H^\pm(x_n, \tau_k, 2\pi / \omega_0)(u, p) = A_{(x_n, \tau_k \pm \delta, p)}(u + i2\pi / p),
\]

then we have the crossing number

\[
\gamma_1(x_n, \tau_k, 2\pi / \omega_0) = \deg_B(H^-(x_n, \tau_k, 2\pi / \omega_0), \Omega_\epsilon) - \deg_B(H^+(x_n, \tau_k, 2\pi / \omega_0), \Omega_\epsilon) = -1.
\]

By Theorem 3.2 of Wu [16], we conclude that the connected component \( C(y_n, \tau_k, 2\pi / \omega_0)\) through \((y_n, \tau_k, 2\pi / \omega_0)\) in \( \Sigma\) is nonempty. Meanwhile, we have

\[
\sum_{(x_n, \tau, p) \in C(y_n, \tau_k, 2\pi / \omega_0)} \gamma_1(x_n, \tau, p) < 0
\]

and hence, by Theorem 3.3 of Wu [16], \( C(y_n, \tau_k, 2\pi / \omega_0)\) is unbounded.

Lemma 4.1 implies that the projection of \( C(x_n, \tau_k, 2\pi / \omega_0)\) onto the \( x\)-space is bounded. Similar to Lemma 4.2, one can get that Eq. (4.1) with \( \tau = 0\) has no nonconstant periodic solutions. Therefore, the projection of \( C(y_n, \tau_k, 2\pi / \omega_0)\) onto the \( \tau\)-space is bounded below. From (2.7), we have

\[
\tau_k = \frac{(1 + y_n^2)}{by_n \sqrt{n^2 - (1 + y_n^2)}} \left[ \cos^{-1} \left( -\frac{(1 + y_n^2)}{n} \right) + 2k\pi \right] , \quad k = 0, 1, \ldots.
\]

Thus, when \( k > 0\) we have \( 2\pi / \omega_0 < \tau_k\). For a contradiction, we suppose that the projection of \( C(y_n, \tau_k, 2\pi / \omega_0)\) onto \( \tau\)-space is bounded. This means that the projection of \( C(y_n, \tau_k, 2\pi / \omega_0)\) onto \( \tau\)-space is included in an interval \((0, \tau^*)\). Noticing \( 2\pi / \omega_0 < \tau_k\) and applying Lemma 4.2, we have \( 0 < p < \tau^*\) for \((y(t), \tau, p)\) belonging to \( C(y_n, \tau_k, 2\pi / \omega_0)\). This implies that the projection of \( C(y_n, \tau_k, 2\pi / \omega_0)\) onto \( p\)-space is bounded and the connected component \( C(y_n, \tau_k, 2\pi / \omega_0)\) is bounded. This contradiction completes the proof. \( \square \)

5. Numerical examples

In this section, we present some numerical results of system (2.1) at different values of \( a, b, \) and \( \tau\). From Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. We first consider the following system:

\[
\dot{y}(t) = \frac{5}{17} - \frac{10y(t)y^4(t - \tau)}{1 + y^4(t - \tau)}, \tag{5.1}
\]

which has a positive equilibrium \( y_\ast = 1/2\) and satisfy the condition \( n > 1 + y_\ast^4\). From the formulae in Section 2, it follows that \( \{\tau_0 = 0.8616, \tau_1 = 3.804, \tau_2 = 6.7476, \tau_3 = 9.6905, \ldots\} \) and \( \{\text{Rec}_1^0(0) = -27.8846, \text{Rec}_1^1(0) = -123.122, \ldots\} \).
Rec^2_1(0) = -218.36, Rec^3_1(0) = -313.598, . . .}, which, together with (3.32), implies μ_2 > 0 and β_2 < 0. Thus, y_* is stable when τ < τ_0 as is illustrated by the computer simulations (see Fig. 1). When τ pass through the critical value τ_0, y_* loses its stability and a Hopf bifurcation occurs, i.e. a family of periodic solutions bifurcates from y_*.

Since μ_2 > 0 and β_2 < 0, the direction of the bifurcation is τ > τ_0, and these bifurcating periodic solutions from y_* at τ_0 are stable, which are depicted in Figs. 2 and 3.

Fig. 1. The equilibrium of (2.1) is asymptotically stable when τ = 0.6 ≤ 0.8616.

Fig. 2. The equilibrium of (2.1) is unstable and bifurcating periodic solution from y_* occurs when τ = 2 > 0.8616.
Fig. 3. $\tau = 6 > 0.8616$, bifurcating periodic solution from $y_*$ occurs.

References