Transient and steady state response (cont.)
Example DC Motor

• Page 111 Ex.1-4-3
Effects of a third pole and a zero on the Second-Order System Response

• For a third-order system with a closed-loop transfer function

\[ T(s) = \frac{1}{(s^2 + 2\zeta s + 1)(\gamma s + 1)} \]

• The s-plane is
Effects of a third pole and a zero on the Second-Order System Response (cont.)

- The third-order system is normalized with $\omega_n=1$.
- The response of a third-order system can be approximated by the dominant roots of the second-order system
  - As long as the real part of the dominant roots is less than 1/10 of the real part of the 3rd root.

\[
\left| \frac{1}{\gamma} \right| \geq 10|\zeta \omega_n|
\]

- **Dominant roots**: The roots of the characteristic equation that represent or dominate the closed-loop transient response
  - Example 3-5-3 estimate the damping ratio
Effects of a third pole and a zero on the Second-Order System Response (cont.)

- If the transfer function of a system possesses finite zeros and they are located relatively near the dominant complex poles, then the zeros will significantly affect the transient response of the system.
- The transient response of a system with one zero and two poles may be affected by the location of the zero.
Effects of a third pole and a zero on the Second-Order System Response (cont.)
Relationship between steady state error and system type

The general form of the open loop transfer function of a system is given by

\[ KG(s)H(s) = \frac{K \prod_{i=1}^{m} (s + z_i)}{s^y \prod_{j=1}^{n} (s + p_j)} \]

Here \( y \) is known as the system type, and it corresponds to the number of integrators in the system.

We can calculate SSE for different types of standard signals

<table>
<thead>
<tr>
<th>( y )</th>
<th>System type</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Type zero</td>
</tr>
<tr>
<td>1</td>
<td>Type 1</td>
</tr>
<tr>
<td>2</td>
<td>Type 2</td>
</tr>
</tbody>
</table>
Example (unitary feedback )

- Page 119, 120, 121
Relationship between steady state error and system type

A) Step Input

\[ R(s) = \frac{R_0}{s} \]

\[
\text{SSE} = \lim_{s \to 0} s \left( \frac{R_0}{s} \right) = \frac{R_0}{s} = \frac{R_0}{1 + G(0)} = \frac{R_0}{1 + K_p}
\]

Here \( K_p \) is known as the position error constant.

\[
\text{For } y > 0
\]

\[
\text{SSE} = \lim_{s \to 0} s^\gamma R_0 \prod_{j=1}^{n} (s + p_j) = \lim_{s \to 0} s^\gamma R_0 \prod_{j=1}^{n} (s + p_j) + K \prod_{i=1}^{m} (s + z_i) = 0!
\]
Relationship between steady state error and system type

For type 0 system...

\[ SSE = \lim_{s \to 0} \frac{s \left( \frac{R_1}{s^2} \right)}{1 + s^y \prod_{j=1}^{n} (s + p_j)} = \lim_{s \to 0} \frac{s^{-1} R_1 \prod_{j=1}^{n} (s + p_j)}{s^y \prod_{j=1}^{n} (s + p_j) + K \prod_{i=1}^{m} (s + z_i)} = \infty \]

For type 0 system the SSE = \infty

For type 1 system...

\[ SSE = \frac{R_1}{K \prod_{i=1}^{m} z_i} = \frac{R_1}{K_c} \]

velocity constant

For \( y > 1 \) the steady state error is zero.
Relationship between steady state error and system type

C) Parabolic Input

\[ R(s) = \frac{R_2}{s^3} \]

\[
SSE = \lim_{s \to 0} \frac{s^3}{s^3} R_2 \prod_{i=1}^{m} (s + z_i) = \lim_{s \to 0} \frac{s^{2R_2} \prod_{j=1}^{n} (s + p_j)}{s^{n} \prod_{j=1}^{n} (s + p_j) + K \prod_{i=1}^{m} (s + z_i)}
\]

For type 0 and type 1 systems, the steady state error is infinite. For type 2 systems the steady state error is given by

\[
SSE = \frac{R_2}{K \prod_{i=1}^{m} z_i} = \frac{R_2}{K_a \prod_{j=1}^{n} p_j}
\]

acceleration error constant
Relationship between steady state error and system type

<table>
<thead>
<tr>
<th>System Type (number of pure integrators)</th>
<th>Input</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>step</td>
</tr>
<tr>
<td>0</td>
<td>$SSE = \frac{R_0}{1 + K_p}$</td>
</tr>
<tr>
<td>1</td>
<td>$SSE = 0$</td>
</tr>
<tr>
<td>2</td>
<td>$SSE = 0$</td>
</tr>
</tbody>
</table>

Conclusions:
1. Adding integrators (increasing system type) eliminates steady state error.
2. If steady state error is finite and not zero, then increasing the system dc gain (increasing controller proportional gain, adding poles near the origin and/or zeros far away from the origin on the LHS of s-plane), reduces steady state error.
Example Problems

Calculate the open and closed loop steady errors, ramp errors, position constants or velocity constants to step and ramp inputs, respectively, for the following cases.

Problem 1

\[ KH(s)G(s) = \frac{10(s + 2)}{(s + 4)(s + 20)} \]

Problem 2

\[ KH(s)G(s) = \frac{10(s + 2)}{s(s + 4)(s + 20)} \]

Problem 3 (two problems)

\[ KH(s)G(s) = \frac{8s \pm 1}{s(s^2 + 3s + 1)(s^2 + 40s + 70)} \]

Problem 4

\[ KH(s)G(s) = \frac{-10s + 2}{(s^2 + 4)(s^2 + 30s + 50)} \]

The steady-state and ramp error computations are only valid if the closed loop is stable. Check which of these examples provides a stable closed loop.
System types

• Consider the unity-feedback control system with following open loop transfer function:

\[
G(s) = \frac{k(s^m + b_{m-1}s^{m-1} + \ldots + b_1s + b_0)}{s^q(s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0)}
\]

• It involves the terms in denominator, representing a pole of multiplicity q at the origin. This present the number of integrators.

• A system is called type 0, type 1, type 2, …if q=0, q=1, q=2, …, respectively.
Example (unitary feedback)

- Look at Table 1-3 on page 126 for a summary of Steady-State Errors

- The power of $s$ on the denominator ($s^q$) denotes the type of system
The Steady-State Error of Feedback Control Systems (importance of feedback)

- The actual system error is $E(s) = R(s) - Y(s)$

$$E(s) = R(s) - \frac{G(s)}{1 + GH(s)} R(s) = \left[1 + GH(s) - G(s)\right] \frac{1}{1 + GH(s)} R(s)$$

When $H(s) = 1 \Rightarrow E(s) = \frac{1}{1 + G(s)} R(s)$

$$\Rightarrow \lim_{t \to \infty} e(t) = e_{ss} = \lim_{t \to 0} \frac{sR(s)}{1 + G(s)}$$
The Steady-State Error of Feedback Control Systems (cont.)

- **Step Input**
  - $A$ is the magnitude of the input

\[
e_{ss} = \lim_{t \to 0} \frac{s(A/s)}{1 + G(s)} = \frac{A}{1 + G(0)}
\]

Position error constant : \( K_p = \lim_{t \to 0} G(s) \)

\[
\Rightarrow e_{ss} = \frac{A}{1 + K_p}
\]

- Figure (22-3)
- Example: page 123, 2-6-3
The Steady-State Error of Feedback Control Systems (cont.)

- **Ramp (velocity) Input**
  - $A$: the slope of the ramp

\[
e_{ss} = \lim_{t \to 0} \frac{s(A/s^2)}{1 + G(s)} = \lim_{t \to 0} \frac{A}{s + sG(s)} = \lim_{t \to 0} \frac{A}{sG(s)}
\]

Velocity error constant : $K_v = \lim_{t \to 0} sG(s)$

\[
\Rightarrow e_{ss} = \frac{A}{(K \prod z_i / \prod p_k)} = \frac{A}{K_v}
\]

- Figure 23-3
The Steady-State Error of Feedback Control Systems (cont.)

- **Acceleration Input**
  - \( r(t) = At^2/2 \)

\[
r(t) = At^2/2
\]

\[
e_{ss} = \lim_{s \to 0} s\left(\frac{A}{s^3}\right) = \lim_{s \to 0} \frac{A}{s^2G(s)}
\]

Accelerative error constant: \( K_a = \lim_{s \to 0} s^2G(s) \)

\[
\Rightarrow e_{ss} = \frac{A}{K \prod z_i / \prod p_k} = \frac{A}{K_a}
\]

- Figure 24-3
Effect of feedback!

• Time constant (rate of system response)
  – How make it smaller or bigger by feedback using!
    • Figures: 25-3, 26-3

• External disturbance
  – Always living in our systems!
  – Two type
    • Load dist. Or offset
    • Random noise
  – Figures 27-3, 28-3
  – SNR: signal-to-noise-ratio

• Sensitivity
Sensitivity of Control Systems to Parameter Variations

• With an open-loop system, all changes and errors at the output are ignored, resulting in a changing and inaccurate output.

• A closed-loop system senses the changes in the output and attempts to correct the output.

• The **sensitivity** of a control system to parameter variations is very important.
  – A closed-loop system can reduce the system’s sensitivity.
Sensitivity of Control Systems to Parameter Variations (cont.)

- For the closed-loop case:
  \[ Y(s) = \frac{G(s)}{1 + GH(s)} R(s) \]
  
  if \( GH(s) \gg 1 \) \( \Rightarrow \) \( Y(s) \approx \frac{1}{H(s)} R(s) \)
  
  - The output is only affected by \( H(s) \).
  - If \( H(s) = 1 \), we have the desired result

- **Caution**: The requirement that \( GH(s) \gg 1 \) may cause the system response to be highly oscillatory and even unstable.

- As we increase the magnitude of the loop transfer function \( G(s)H(s) \), we reduce the effect of \( G(s) \) on the output.

- The first advantage of a feedback system is that the effect of the variations of the process, \( G(s) \), is reduced.
Sensitivity of Control Systems to Parameter Variations (cont.)

• Illustration of the parameter variations
  – Let’s consider a change in the process so that the new process is $G(s)+\Delta G(s)$.
  – The change in the transform of the output is

Open - Loop System

$\Delta Y(s) = \Delta G(s)R(s)$

Closed - Loop System

$Y(s)+\Delta Y(s) = \frac{G(s)+\Delta G(s)}{1+(G(s)+\Delta G(s))H(s)}R(s)$

$\Delta Y(s) = \frac{\Delta G(s)}{(1+GH(s)+\Delta GH(s))(1+GH(s))}R(s)$

When $GH(s) >> \Delta GH(s)$

$\Delta Y(s) = \frac{\Delta G(s)}{[1+GH(s)]^2}R(s)$
Sensitivity of Control Systems to Parameter Variations (cont.)

• For the closed-loop system
  – The change in the output of the closed-loop system is reduced by the factor \([1+GH(s)]\)
  – This is usually much greater than 1 in the range of complex frequencies of interest.
Sensitivity of Control Systems to Parameter Variations (cont.)

• The system sensitivity

\[
\text{System Sensitivity } y = \frac{\text{percentage change in the system transfer function}}{\text{percentage change of the process transfer function}}
\]

- System Transfer Function is

\[
T(s) = \frac{Y(s)}{R(s)}
\]

System Sensitivity \( y = S = \frac{\Delta T(s)/T(s)}{\Delta G(s)/G(s)} \)

• For small incremental changes:

\[
S = \frac{\partial T / T}{\partial G / G} = \frac{\partial \ln T}{\partial \ln G}
\]
Sensitivity of Control Systems to Parameter Variations (cont.)

- The system transfer function of the closed-loop system is:

\[ T(s) = \frac{G(s)}{1 + GH(s)} \]

- The sensitivity of the feedback system is:

\[ S_G^T = \frac{\partial T}{\partial G} \frac{G}{T} = \frac{1}{(1 + GH)^2} \frac{G}{G / (1 + GH)} \]

\[ S_G^T = \frac{1}{1 + G(s)H(s)} \]

- The sensitivity of a system may be reduced below that of the open-loop system by increasing GH(s) over the frequency range of interest.
Sensitivity of Control Systems to Parameter Variations (cont.)

• The sensitivity of the feedback system to changes in the feedback element $H(s)$ is:

$$S_H^T = \frac{\partial T}{\partial H} \frac{H}{T} = \left( \frac{G}{1 + GH} \right)^2 \frac{-H}{G/(1 + GH)}$$

$$S_H^T = \frac{-GH}{1 + GH}$$

– When $GH$ is large, the sensitivity approaches unity $(1)$

$$S_H^T \approx 1$$

– The changes in $H(s)$ directly affect the output response

• It is important to use feedback components that will not vary with environmental changes or that can be maintained constant.
Sensitivity of Control Systems to Parameter Variations (cont.)

- The sensitivity to $\alpha$:

$$S^T_\alpha = S^T_G S^G_\alpha$$

$\alpha$ : a parameter within the transfer function of the block $G$

- The transfer function of the system $T(s)$ is a fraction of the form

$$T(s, \alpha) = \frac{N(s, \alpha)}{D(s, \alpha)}$$

$\alpha$ : a parameter that may be subject to variation due to the environment

- The sensitivity to $\alpha$:

$$S^T_\alpha = \left. \frac{\partial \ln T}{\partial \ln \alpha} \right|_{\alpha_0} - \left. \frac{\partial \ln D}{\partial \ln \alpha} \right|_{\alpha_0} = S^N_\alpha - S^D_\alpha$$

  - $\alpha$ is a nominal value of the parameter
Sensitivity of Control Systems to Parameter Variations (cont.)

- $K_a$: gain of amplifier
- Output voltage:

$$v_0 = -K_a \cdot v_{in}$$

- We add feedback using a potentiometer $R_p$.

- The transfer function of the amplifier without feedback is: 

$$T = -K_a$$

- The sensitivity to changes is the amplifier gain is:

$$S_{T/K_a} = 1$$

• Example: 1-7-3
STABILITY
Examining the closed loop poles

The zeros of the denominator

\[ 1 + KG_c G_p H = 0 \]

will determine the impulse and step response stability.

The inverse Laplace transform applied to

\[ \frac{b}{s - p} \]

gives time signal

\[ be^{at} = be^{(a+ jb)t} = be^{at} (\cos bt + j \sin bt) \]

For stability all closed loop poles must have negative real parts.
Ruth-Hurwitz Criterion Procedure

• The characteristic equation in the Laplace variable is:

\[ q(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 = 0 \]

In factored form:

\[ a_n (s - r_1)(s - r_2)...(s - r_n) = 0 \]

Multiplying the factors together:

\[ q(s) = a_n s^n - a_n (r_1 + r_2 + ... + r_n) s^{n-1} \]
\[ + a_n (r_1 r_2 + r_2 r_3 + r_1 r_3 ... s^{n-2} \]
\[ - a_n (r_1 r_2 r_3 + r_1 r_2 r_4 + ... s^{n-3} + ... \]
\[ + a_n (-1)^n r_1 r_2 r_3 ... r_n = 0 \]
Ruth-Hurwitz Criterion Procedure

- For the nth-degree equation:

\[ q(s) = a_n s^n - a_n \left( \text{sum of all roots} \right) s^{n-1} \]
\[ + a_n \left( \text{sum of the products of the roots taken 2 at a time} \right) s^{n-2} \]
\[ - a_n \left( \text{sum of the products of the roots taken 3 at a time} \right) s^{n-3} + \ldots \]
\[ + a_n (-1)^n \left( \text{product of all n roots} \right) = 0 \]

- Note that all the coefficients of the polynomial must have the same sign if all the roots are in the left-hand plane.
- For a stable system all the coefficients must be nonzero.
- Both of these requirements must be sufficient for the system to be stable.
- If the are satisfied we can proceed to check for other conditions to prove that the system is stable.
Ruth-Hurwitz Criterion Procedure

Does this equation have only stable roots? 

\[ a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_o = 0 \]

(I.e do all solutions have negative real parts? )

For first and second order systems (n=1,2) the necessary and sufficient condition for stability is that the coefficients of the polynomial are non-zero and all have the same sign.

For higher order systems produce this table

\[
\begin{array}{cccccc}
  s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
  s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
  s^{n-2} & c_1 & c_2 & c_3 & \ldots \\
  s^{n-3} & d_1 & d_2 & d_3 & \ldots \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  s^1 & j_1 & & & & \\
  s^0 & k_1 & & & & \\
\end{array}
\]

\[
\begin{align*}
  c_1 &= \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \\
  c_2 &= \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \\
  c_3 &= \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}} \\
  d_1 &= \frac{c_1 a_{n-3} - a_{n-1} c_2}{c_1} \\
  d_2 &= \frac{c_1 a_{n-5} - a_{n-1} c_3}{c_1} \\
  \ldots &= \ldots \\
\end{align*}
\]
Ruth-Hurwitz Criterion Procedure

Case 2: there is a zero in the first column but that row is not zero everywhere

Replace the 0 by a small $\varepsilon > 0$ positive and carry on in the usual way

Again the number of sign changes in the first column will determine the number of unstable zeros.

1. Product polynomial at (s+1) start roth table
2. Replace s with 1/x
Ruth-Hurwitz Criterion Procedure

Case 2: there is a zero in the first column and that row is zero everywhere.

The auxiliary polynomial (order is always even) gives the number of symmetrical root pairs (to the origin).

It divides the polynomial and long division can be used to obtain the other factor.

For the rest of this class practice the application of the Ruth-Hurwitz procedure to decide on stability for as many of the examples as you can.

(The solutions involve the computation of the closed loop transfer functions first.)
Examples: Ruth-Hurwitz criterion procedure

Example 2

\[ G(s) = \frac{K}{s^4 + 3s^3 + 3s^2 + 2s + K} \]

Condition of stability:

\[ 0 < K < \frac{14}{9} \]
Examples: Ruth-Hurwitz criterion procedure

Example 3: zero in first column (but not all zero in row)

\[ s^4 + s^3 + 2s^2 + 2s + 5 = 0 \]

\[
\begin{array}{ccc}
  s^4 & 1 & 2 & 5 \\
  s^3 & 1 & 2 & 0 \\
  s^2 & \varepsilon & 5 & 0 \\
  s^1 & \frac{2-5}{\varepsilon} & 0 \\
  s^0 & 5 \\
\end{array}
\]

small positive number

large negative number
Examples: Ruth-Hurwitz criterion procedure

Example 4: a zero row

\[ s^4 + 2s^3 + 11s^3 + 18s + 18 = 0 \]

Coefficients for auxiliary equation

\[
\begin{array}{ccc}
 s^4 & 1 & 11 \\
 s^3 & 2 & 18 \\
 s^2 & 2 & 18 \\
 s^1 & 0 & 0 \\
 s^0 & & \\
\end{array}
\]

Long division of original polynomial by auxiliary polynomial:

\[
\frac{s^4 + 2s^3 + 11s^2 + 18s + 18}{s^4 + 2s^3 + 9s^2 + 18s} = s^2 + 9
\]

\[
\begin{array}{c|cc}
 s^4 & 2s^3 & 11s^2 \\
 \hline
 s^4 & 2s^3 & 9s^2 \\
 & & 18s \\
 & & 18 \\
 & & 2s \\
 & & 2 \\
\end{array}
\]

\[
= (s^2 + 9)(s^2 + 2s + 2)
\]

Conclusion: unstable