Technical Communique

A remark on passivity-based and discontinuous control of uncertain nonlinear systems

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Abstract

We address the problem of robust stabilisation of nonlinear systems affected by time-varying uniformly bounded (in time) affine perturbations. Our approach relies on the combination of sliding mode techniques and passivity-based control. Roughly speaking we show that under suitable conditions the sliding mode variable can be chosen as the passive output of the perturbed system. Then we show how to construct a controller which guarantees the global uniform convergence of the plant’s outputs towards a time-varying desired reference, even in the presence of permanently exciting time-varying disturbances. We illustrate our result on the tracking control of the van der Pol oscillator. © 2001 Published by Elsevier Science Ltd.

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1. Introduction

Motivated by real life applications, robust adaptive control has been a subject of study for many years now. Numerous approaches, sometimes based on the physical structure of the system, have been proposed using techniques as backstepping (Freeman & Kokotovic, 1996), sliding modes (Utkin, 1992; Sira-Ramirez, 1993), passivity-based (Byrnes, Isidori, & Willems, 1991; Ortega, Loría, Nicklasson, & Sira-Ramirez, 1998), feedback linearisation (Isidori, 1995; Nijmeijer & van der Schaft, 1990) or high gain (Marino & Tomei, 1993).

From a theoretical viewpoint, sliding mode control results are very attractive since ideally, they exactly compensate for disturbances (which are bounded in time) and allow convergence in finite time, to the so-called sliding manifold. Then, if the zero-dynamics corresponding to this ideal sliding motion is asymptotically stable, perfect tracking is achieved. The fact that the “disturbances” may represent unmodeled (actuator) dynamics or nonlinearities due to state estimation mismatch in output feedback control (Emelyanov, Korovin, Nersisian, & Nizenson, 1992; Sanchis & Nijmeijer, 1998), makes the field of application of this approach potentially very wide. Even though sliding mode control induces an often undesirable chattering in the input which uses the actuators, a popular area of application is in tracking control of mechanical and electromechanical systems (Stepanenko & Yuan, 1993; Slotine & Sastry, 1983; Gorez, 1999; Panteley, Ortega, & Gafvert, 1998). To avoid chattering often precautions as for instance the introduction of a boundary layer are taken, cf. Utkin (1992).

In this communique, we highlight some links between two popular control approaches: passivity based (Byrnes et al., 1991; Ortega et al., 1998) and sliding mode. The advantage of the former is that it is an energy-based approach and therefore, it exploits the known physical structure of the system. As pointed out above, discontinuous controls are (theoretically) attractive in robust control applications. Typically, for relative-degree-one systems the control design is based on first, tracking the
system trajectories to a sliding manifold, and then, in designing a control law which keeps the trajectories in it. The usual criteria considered to design the sliding surface is that it should define an asymptotically stable zero-dynamics. However, in general, there is no engineering intuition on how to design a suitable sliding surface satisfying the required conditions mentioned.

In this note, we will show that robust control of uncertain systems with affine disturbances, may be possible when the given nominal system is passifiable via state feedback and then, under appropriate assumptions on the disturbances, a discontinuous term is included in the controller as to compensate for the disturbances. Even though this technique is well understood in the literature of sliding modes, the novelty in our result is to look at it from a passivity viewpoint. Indeed, our proof relies on a passivity approach rather than on a classical Lyapunov analysis. To put our discussion in perspective and to introduce some notation, let us briefly recall some aspects from passivity-based and discontinuous control.

2. Preliminaries

2.1. Passivity-based control

Consider the affine nonlinear system
\[ \dot{x} = f(x) + g(x)u, \]
\[ y := h(x), \tag{1} \]
where \( x \in \mathbb{R}^n, y \in \mathbb{R}, f, g \) and \( h \) are continuous vector fields, respectively functions of \( x \). Assume further that system (1) defines a passive operator \( \Sigma: u \mapsto y \) with a \( \mathcal{F}^1 \) storage function \( V := V(x) \) which is bounded from below. Under these considerations, it is well known (Jurdjević & Quinn, 1978) that the control law \( u = u^* \) with \( u^* = - (\partial V/\partial x) g(x) \) asymptotically stabilises the origin of (1). A passivity interpretation of the Jurdjević and Quinn controller \( u^* \) is contained in the results of Theorem 3.2 of Byrnes et al. (1991).

**Theorem 1.** If the system (1) is passive and zero-state detectable\(^1\) then the controller
\[ u = - \phi(y), \]
with \( \phi(0) = 0 \) and \( \phi(y) > 0 \) for all \( y \neq 0 \), asymptotically stabilises the origin \( x = 0 \). Further, if the storage function \( V(x) \) is proper then the origin is globally asymptotically stable.

\(^1\) We remind the reader that a system with output \( y(t) = h(x(t)) \) is zero-state detectable if \( y(t) \equiv 0 \) implies that \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

2.2. Discontinuous control to compensate for bounded disturbances

We briefly recall in this section a well known technique based on discontinuous control. For a more detailed exposition see e.g. Khalil (1996, Chapter 13). For our purposes let us consider the following simple example of a scalar affine system
\[ \dot{x} = f(x) + d(t) + u, \tag{2a} \]
\[ y = h(x), \tag{2b} \]
where \( f(x) \) is locally Lipschitz continuous, \( f(0) = 0, x \in \mathbb{R} \) and assume that there exists an unknown positive constant \( \delta \) such that \( |d(t)| < \delta \) uniformly in \( t \). Assume further that in the absence of the disturbance (that is, if \( d \equiv 0 \)) a controller \( u = u^*(x) \) is known such that the system \( \dot{x} = f(x) + u^*(x) \) is globally asymptotically stable (GAS). From the GAS assumption it follows (see e.g. Hahn (1967)) that there exists a Lyapunov function \( V = V^*(x) \), radially unbounded, such that
\[ \dot{V}^* = \frac{\partial V^*}{\partial x}[f(x) + u^*(x)] \leq - \sigma(|x|), \tag{3} \]
where \( \sigma \in \mathcal{K} \) (a function \( \sigma: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} \) is of class \( \mathcal{K} \), \( \sigma \in \mathcal{K} \)) if it is continuous, strictly increasing and zero at zero). Then a well known result in the literature of sliding mode control (see e.g. Utkin, 1992; Sira-Ramirez, 1993) is the following. Define \( \text{sgn}(\sigma) = -1 \) if \( \sigma < 0 \), \( \text{sgn}(\sigma) = 1 \) if \( \sigma > 0 \), and \( \text{sgn}(0) \in [-1,1] \), and the sliding surface
\[ \sigma(x) := - \frac{\partial V^*(x)}{\partial x}, \tag{4} \]
then the adaptive control law
\[ u = u^*(x) - \delta \sigma, \tag{5} \]
\[ \dot{\delta} = \text{sgn}(\delta) \sigma, \tag{6} \]
renders the system (2) globally convergent, that is, for any initial conditions \( x(t_0) \) we have that
\[ \lim_{t \to \infty} x(t) = 0. \tag{7} \]
For the sake of completeness, we proceed to prove the claim above. As a matter of fact, the proof of our main result will follow along similar lines. First, notice that since \( \delta \) is constant the closed loop system (2), (5) is given by
\[ \dot{x} = f(x) + u^*(x) - \delta \sigma \text{sgn}(\sigma) + d(t), \tag{7} \]
\[ \dot{\delta} = \text{sgn}(\delta) \sigma, \tag{8} \]
where we defined the estimation error \( \dot{\delta} := \delta - \delta \). Define also the Lyapunov function \( V(x, \delta) := V^*(x) + \frac{\delta^2}{2} \), then we proceed with calculating the time derivative of the latter along the trajectories of the closed loop system (7), (8) which is defined by the differential inclusion \( \dot{x} \in F(t, x) \)
where \( \chi := \text{col}[x,d] \) and
\[
F(t,\chi) := \begin{cases} \begin{pmatrix} (7),(8) \\ f(x) + u^*(x) - \delta \dot{\chi} + d(t) \end{pmatrix} & \sigma = 0, \\ 0 & \sigma \neq 0, \end{cases}
\]
where \( \lambda \in [-1,1] \). Thus, using \( |d(t)| < \delta \) and \( |\sigma| = \text{sgn}(\sigma)\sigma \), the time derivative of \( V \) satisfies
\[
\dot{V}(x,\delta) = V^*(x) - (\delta - \delta)\left(|\sigma| - \frac{\partial V^*(x)}{\partial x}\right),
\]
for all solutions \( \chi(t) \) such that \( \dot{\chi} \in F(t,\chi(t)) \) and where \( V^*(x) \) is defined in (3). Using (4), it is clear that the addition of the last two terms on the right hand side of (10) is exactly zero, hence from the assumption that the unperturbed closed loop system \( \dot{x} = f(x) + u^*(x) \) is GAS we obtain that
\[
\dot{V}(x,\delta) \leq -\alpha(|x|).
\]
The global convergence of \( x(t) \) follows using Barbalat’s lemma and standard arguments (see e.g. Desoer & Vidyasagar, 1975). Moreover, notice that in the case when \( \delta \) is known, GAS follows using standard Lyapunov arguments for systems with discontinuous right hand sides (see e.g. Clarke, Ledyaev, Stern, and Wolenski (1998)). The following conclusions are useful at this point:

1. Assume that system (2a) with zero disturbance defines an output strictly passive operator \( \Sigma : u \mapsto y \) with respect to the output
\[
y = \frac{\partial V^*(x)}{\partial x},
\]
and assume further that system (2a) is zero-state detectable with respect to the same output \( y \). Under these conditions one can construct a globally stabilising control law \( u = u^*(x) \) using Theorem 1.

2. If we substitute the output definition (11) in the controller (5), (6) then the sliding variable happens to be the output of the unperturbed system for which a globally stabilising controller is available. From this, one might conjecture that the controller (5), (6), (11) guarantees the global convergence of \( x(t) \) under the assumptions that (2) defines an output strictly passive operator (OSP) \( u + d(t) \mapsto y \) and a “detectability condition”. The formal statement of this conjecture is strictly contained in our main result which we present below.

3. Main result

We consider nonlinear systems perturbed by affine disturbances, with control input \( u \in \mathbb{R}^m \) and output \( y \in \mathbb{R}^n \),
\[
\dot{x} = f(t,x) + g(t,x)[u + d(t,x)],
\]
(12a)
\[
y = h(t,x),
\]
(12b)
where the fields \( f(t,x), g(t,x) \), and the function \( h(t,x) \) are locally Lipschitz continuous, the input and the disturbance \( d : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}^n \) satisfies the following:

A 1 Each component of the vector field \( d \) is uniformly bounded in \( t \) as follows
\[
|d_i(t,x)| \leq \theta_1 \delta_i(|x|) + \theta_2, \quad \forall i \leq n,
\]
where \( \theta_1 \) and \( \theta_2 \) are unknown non-negative constants and \( \delta_i(|x|) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0} \) is a known continuous function.

An obvious implication of (13) is the existence of a continuous function \( \delta : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0} \) and positive constants \( \theta_1 \) and \( \theta_2 \) such that \( |d(t,x)| \leq \theta_1 \delta(|x|) + \theta_2 \), or in compact form
\[
|d(t,x)| \leq \theta^T A(|x|),
\]
(14)
where we have defined \( \theta^T := (\theta_1, \theta_2) \) and \( A(|x|) := \delta(|x|),1 \).

It is important to remark that the disturbance \( d(t,x) \) may be the result of unmodelled dynamics, noisy measurements, parameter uncertainty or, thinking of the system (12) with \( d \equiv 0 \) as a passive system (\( u \mapsto y \)), the disturbances \( d \) are non dissipative forces affecting the plant. Physical systems modeled by (12) are numerous, e.g. Euler–Lagrange systems or chaotic oscillators such as the periodically forced Duffing equation or the van der Pol system and certain under-actuated systems. We are interested in a design methodology for a controller \( u = u(t,x) \) which guarantees that the solution \( x(t; t_0, x_0) \) of (12) with initial conditions \( t_0, x_0 = x(t_0) \) is globally convergent, that is, \( \lim_{t \to \infty} x(t) = 0 \) for any initial conditions \( (t_0, x_0) \).

To that end we will use state discontinuous controllers like (5). As previously discussed, an important implication of this is that the usual regularity conditions for unicity of (closed loop) solutions do not hold. We will not go deeper into this question but refer the reader to Clarke et al. (1998). We will simply notice that for the purposes of this note it is sufficient to observe that the only discontinuity in the closed loop dynamics is due to \( \text{sgn}(\cdot) \) which implies the existence of (limiting) solutions and that they exist for all time, that is, the closed loop system is forward complete (see Clarke et al. (1998)).

This said, let us introduce the following definition which we will use in order to provide a “passivity-based” proof of our main result.

**Definition 1** (Strong zero-state detectability). Consider the system \( \dot{x} = f(t,x) + v \) where \( f(\cdot,\cdot) \) is such that all existing solutions \( x(\cdot, t, x) \) are defined on \( [t, \infty) \). Consider also the closed set \( \mathcal{U} \subset \mathbb{R}^n \). The system, with inputs
where \( |y| \) stands for the Frobenius norm, i.e. \( |y| := \sum_i |y_i| \).

Notice that the subtraction of the last two terms on the right hand side of (21) is non-positive hence

\[
\dot{V}_{(12,16,17)}(t, x, \tilde{\theta}) \leq -\beta|y|^2 \leq 0,
\]

hence the system is uniformly globally stable. The inequality (22) also implies that all \( V(t, x(t), \tilde{\theta}(t)) \) are decreasing and hence bounded so the complete state \( (x, \tilde{\theta}) \in \mathcal{L}_w^{n+2} \). Furthermore, integrating (22) from \( t_0 \) to \( \infty \) one concludes that \( y \in \mathcal{L}_w^n \). Next, notice from (19), (14) that since \( \tilde{\theta} \) is constant and \( \text{det} \cdot \) is continuous, \( v(t) \) is also uniformly bounded. From the continuity of \( h(t, x) \) it follows that all \( y \in \mathcal{L}_w^n \). Then, from the closed loop dynamics (15a), (19) we obtain that \( x \in \mathcal{L}_w^n \). Hence our control design is passivity-based for all bounded signals \( v(t, x(t)) \) defined in (19) with \( y \equiv 0 \). \( \Box \)

4. Discussion

(a) From a Lyapunov stabilisation viewpoint, our controller \( u(t, x) \) which is designed to render OSP the nominal system, may be compared to the “equivalent control” of the sliding mode approach. However, in contrast to the latter technique, our control design is passivity-based and therefore, exploits the natural structure of the system instead of forcing the trajectories to remain on a sliding manifold for which, in general, there is no physical motivation behind. In our interpretation, the sliding manifold \( \sigma(x) = 0 \) corresponds to zeroing the passive output. However, notice that \( u^w(t, x) \) is not designed with the aim at achieving finite time convergence to the sliding manifold but asymptotic convergence of the passive output. This has important consequences since a known practical disadvantage of discontinuous controllers achieving finite time convergence is chattering.

(b) Some other interesting remarks concerning conditions under which the sliding variable \( \sigma(x) \) can be considered as a passive output are found in Sira-Ramirez (1999). In this reference, sufficient conditions are given to transform an affine nonlinear system into a Hamiltonian one. Then conditions on the resulting dynamics are given so that the system defines a passive map with output \( y = \sigma(x) \) and storage \( \dot{V}^w(t, x) = 0.5\sigma(x)^2 \).

(c) It is often assumed in the literature, that the zero-dynamics defined by \( \sigma(x) = 0 \) are asymptotically stable. Here, we simply assume that the system is strongly zero state-detectable which is a milder condition. In particular notice that if there exists \( x \in \mathcal{X} \) such that \( \sigma(x) \geq \varepsilon(|x|) \) then, convergence of \( x(t) \) follows directly from (22). See e.g. Khalil (1996, Chapter 13) where such condition is used in the form \( \dot{V}^w(t, x) \leq -\varepsilon(|x|) \) instead of our OSP.
condition $\dot{V}^*(t, x) \leq -y^2$. We shall add however, that even though the former is more restrictive, it is met in many applications of mechanical systems, see the van der Pol example below.

(d) Finally, we remark that in Theorem 2 we have implicitly assumed that all parameters of our system are known, however, notice that the perturbation $d(t, x)$ may also contain terms which come from uncertainties in the model. More precisely, consider the nonlinear non-autonomous system

$$\dot{x} = f(t, x, \theta) + g(t, x)u,$$

$$y = h(t, x),$$

where $x \in \mathbb{R}^n$, and $\theta \in \mathbb{R}^l$ of constant unknown parameters. Using Theorem 2 a controller $u = u(t, x)$ guaranteeing that $x(t) \to 0$ as $t \to \infty$ can be designed if the dynamics (23) admit the following parameterisation

$$\dot{x} = f'(t, x) + g(t, x)[u + d(t, x, \theta)],$$

where the nonlinearity $d(t, x, \theta)$ satisfies

$$||d(t, x, \theta)|| \leq \theta_1 A(||x||) + \theta_2$$

where $\theta_1$ and $\theta_2$ are unknown constants and $A(\cdot)$ is a continuous known function.

Notice furthermore, that the parameters vector $\theta$ does not need to be constant, as long as it defines continuous uniformly bounded signals, that is, if $\theta(t) \in L^\infty$ and a bound like (25) be determined.

4.1. Example: the van der Pol oscillator

The van der Pol oscillator is known to be the model of an LC electrical network with a nonlinear load and all elements connected in parallel (see e.g. Kapitaniak (1996)). If we assume that the load is an active circuit with input voltage $v$ and input current $i = \rho(v)$ with $\rho(t) = - v + \frac{1}{4}v^3$. Assuming further that the circuit is excited by an AC current source of frequency $\omega$ and amplitude $Q := q\alpha$, with $q > 0$ we can express the dynamic equations of the circuit by the well known equation

$$\ddot{v} + \mu(1 - v^2)\dot{v} + v = q \cos(\omega t),$$

which van der Pol used to model an electrical circuit with a triode valve. We remark that for the particular case when $\mu = 5, q = 5$ and $\omega = 2.463$ the van der Pol oscillator exhibits a chaotic behaviour (Parlitz & Lauterborn, 1987). Consider now the controlled unperturbed van der Pol equation with the voltage $v$ delivered to the active load as the output to be controlled, that is

$$\ddot{v} + \mu(1 - v^2)\dot{v} + v = u^* + w,$$

where $w$ is a bounded external input which includes the term $q \cos(\omega t)$ and

$$u^*(v) = - k_p \dot{v} - k_d v + \mu(2v \dot{v} - v_d^2 \dot{v}) + v_d,$$

where $v_d$ is the desired constant output voltage and $\dot{v} = v - v_d$. Then, the closed loop system (27), (28) defines an output strictly passive operator $\Sigma: w \mapsto y$ with respect to the output $y = \hat{v} + \alpha \dot{v}$ where $\alpha > 0$ is small enough to ensure that the storage function

$$V(v, \dot{v}) = \frac{1}{2} \dot{v}^2 + \frac{1}{2}(k_p + 1)\dot{v}^2$$

$$\quad + \alpha \dot{v} + \frac{1}{2}(\hat{d} - \mu)\dot{v}^2 + \frac{\alpha^2}{4},$$

is positive definite. Furthermore, by choosing $k_d \geq 2 + \epsilon + \mu$ and $k_p \geq 2\epsilon$ the time derivative of $V(v, \dot{v})$ yields $\dot{V}(v, \dot{v}) \leq - y^2 + \alpha \dot{y}$. The OSP property follows by integrating the latter inequality on both sides, from $0$ to $t$. Hence assumption A2 is met. Secondly, the disturbance satisfies $q \cos(\omega t) \leq \alpha$ so assumption A1 is met with $\theta_1 = q$ and $\theta_2 = 0$. Finally, the strong zero-state detectability condition is met as well observing that the output equation $\dot{v} + \alpha \dot{v} = y$ constitutes a strictly proper and stable filter with input $y$ hence $v \to 0$ as $t \to \infty$ if $y \equiv 0$, regardless of $w(t)$. Thus, using Theorem 2 we can easily construct an adaptive robust control law for the controlled and perturbed van der Pol oscillator for the case where none of the parameters $\omega, \mu$ nor $q$ is known. Notice however, that in the case of all known parameters due to the definition of the passive output one can conclude UGAS.

We have performed some simulations in SIMULINK™ of MATLAB™ to illustrate the performance of the controller proposed above. The parameters used are as follows: $q = 1, \omega = 2.463, \mu = 5, k_p = 20, k_d = 10, \gamma = 0.1$ and a constant reference $v_d = 2$. We performed two tests: in Fig. 1 we illustrate the system’s response and the control input, as well as the external disturbance $q \cos(\omega t)$. Notice that the controller converges to a sinusoid to compensate for the unknown disturbance (the observed offset of 2 units corresponds to the term $v_d = 2$ present in $u$ even when the state errors are zero, see Eq. (28)). In the second simulation (cf. Fig. 2), we tested the controller including white measurement noise. Notice that even though the performance is still quite good, the measurement noise causes a small steady state error. In both cases, the parameters $q$ and $\omega$ were supposed unknown and an adaptive law was used.

Remark 2 (Robot sliding mode control). Starting probably with Slotine and Li (1988) for robot manipulators, the choice $y = \dot{v} + \alpha \dot{v}$ as sliding variable is often used in the literature. This has however, some practical drawbacks. As it was observed in Gorez (1999) a small $\epsilon$ leads to a slow convergence (at least locally) of the tracking error $e(t)$ to zero while a large $\epsilon$ induces undesirably high velocities in the sliding motion. In this respect, it is important to remark that on one hand, the passivity properties are invariant with respect to the normalized
output $y = \dot{v} + e\dot{\delta}$ with $e := e / (1 + ||\delta||)$, on the other hand, this output was found in experiments to perform best in robot control applications (see Gorez (1999) and references therein).

5. Concluding remarks

We have highlighted some links between passivity-based, and discontinuous control design for uncertain nonlinear systems. Most importantly, we have provided with a different insight to the sliding mode technique. The main difference with the common sliding mode controllers is that the control design starts at a passivation step and not at the definition of a sliding regime. The second part of the control design is made with the aim at compensating for the disturbances and not with the aim at tightening the dynamics of the system to a manifold.

**References**


