A Survey on Singularly Perturbed Systems

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Abstract:

Multiple time-scale phenomena are almost unavoidable in real systems and the singular perturbation approach has proven to be a powerful tool for system analysis and control design. Singular perturbation theory has received a lot of attention in the past decades.

In this article, first the concept of perturbation and singular perturbation are studied. Then the related theories and analysis followed by some chosen examples are given in some details. Note that discussions given in the solution of these examples are mainly according to my personal understanding and perception and therefore it might have some deficiencies and imperfections. At the next chapter, the recent work in around 30 journal papers have been categorized due to their subject. This chapter will addresses those who want to research in this area to the useful papers. Moreover, for some of these papers, the concept of the proposed methods is discussed. In section 2.3, some applications regard to singular perturbation problem is mentioned. Finally a typical example has been simulated and the obtained results are discussed.
**Introduction:**

Exact closed-form analytic solutions of nonlinear differential equations are possible only for a limited number of special classes of differential equations. Therefore we have to resort to approximate methods.

Suppose we are given the state equation:

\[ x^* = f(x, t, \epsilon) \]

Where \( \epsilon \) is a small scalar parameter and, under certain conditions, the equation has an exact solution \( x(t, \epsilon) \). We want to find an approximate solution \( \tilde{x}(t, \epsilon) \) such that the approximation error \( x(t, \epsilon) - \tilde{x}(t, \epsilon) \) is negligible for small \( |\epsilon| \). Moreover it is evident that to make our original problem easier, the approximated solution \( \tilde{x}(t, \epsilon) \) must be expressed in terms of equations simpler than the original equation. Quite often, the solution of the state equation exhibits the phenomenon that some variables move in time faster than other variables, leading to the classification of variables as “slow” and “fast”. If the state equations depend smoothly on parameter \( \epsilon \), then the approximated method can be achieved through the regular perturbation and averaging method, while in the singular perturbation problem, we face a more difficult perturbation problem characterized by discontinuous dependence of system properties on the perturbation parameter \( \epsilon \), which involves more coupling between the fast and slow modes of system.
CHAPTER 1:
Singular Perturbation Method:

1.1 Introducing the Standard Singular Perturbation model:

Let $x$ denote the “slow” variables and $z$ denote the fast ones, then the standard singular perturbation problem are stated as follows:

\[
\begin{align*}
x^* &= f(t,x,z,\varepsilon), \\
\varepsilon \times z^* &= g(t,x,z,\varepsilon),
\end{align*}
\]

\begin{align*}
x \in \mathbb{R}^n \\
z \in \mathbb{R}^m
\end{align*} \tag{1.1}

We assume $f$ and $g$ are continuously differentiable in their arguments.

We see that setting $\varepsilon$ to zero, causes an abrupt change in dynamic properties of the system and the dimensions of state equations reduces from $n+m$ to $n$. Thus, the differential equation (1.2) will change to:

\[
0 = g(t,x,z,\varepsilon) \tag{1.3}
\]

Let us show the roots of this equation by

\[
z = h_i(t,x), \quad i = 1, ..., k \tag{1.4}
\]

With assumption of isolated real roots, we have $k$ reduced model that each one corresponds to the related root. If we substitute the (1.4) into (1.1), we have:

\[
\dot{x} = f(t,x,h_i(t,x),0) \tag{1.5}
\]

Since a small value of $\varepsilon$ leads to the fast convergence of $z$ to a root of (1.3) (which is the equilibrium of (1.2).), this equation is sometimes called a quasi-steady-state model. On the other hand, since (1.5) deals with the slow variables $x$, this model is also known as the slow model.
1.2 Singular Perturbation Modelling

Converting physical model into the standard singular perturbation problem is not usually easy. The choice of state variables and choice of $\varepsilon$ require a careful thought. Sometimes we can model the parasitic terms like small time constants, masses, capacitors… through singular perturbation form. These mentioned terms are usually ignored in the simplified model. Thus singular perturbation provides us a tool for modelling these extra terms in some over-simplified models. Let us analyze some chosen examples (from [1]) to see how to model a problem into singular perturbation form.

Example 1.1 An armature-controlled DC motor, shown in figure below, can be modelled by the second-order state equation:

\[
\begin{align*}
J \frac{d\omega}{dt} &= ki \\
L \frac{di}{dt} &= -k\omega - Ri + u
\end{align*}
\]

![Figure 1.1 Armature-controlled DC motor](image)

Where $I$, $u$, $R$ and $L$ are the armature current, voltage, resistance, and inductance, $J$ is the moment of inertia, $w$ is the angular speed, and $ki$ and $kw$ are, respectively, the torque and the back e.m.f developed with constant excitation flux $\Phi$.

Next step is look for the choice of fast variables; since typically $L$ is small, it can play the role of $\varepsilon$ and so the current $i$ account for the fast variable.
In singular perturbation modelling it is preferable to have dimensionless state variables and perturbation parameters. Therefore it is suitable to normalize the above system equations. The new system equations can be represented as:

\[
T_m \frac{dw_r}{dt} = i_r \\
T_e \frac{di_r}{dt} = -\omega_r - i_r + u_r
\]

Where \( T_m \) and \( T_e \) are mechanical and electrical time constants respectively.

In the above equations the state variables are dimensionless but the perturbation parameter \( T_e \) (\( T_e << T_m \)) is still have dimension. It can be seen that if we let \( T_m \) to be the time unit (i.e., \( T_m = \frac{t}{T_m} \)), then \( \frac{T_e}{T_m} \) is a dimensionless parameter which can play the role of \( \varepsilon \). Therefore we can rewrite the normalized system equations in the following form:

\[
\frac{d\omega_r}{dt_r} = i_r \\
\frac{T_e}{T_m} \frac{di_r}{dt} = -\omega_r - i_r + u_r
\]

Now assuming \( \frac{T_e}{T_m} \) as \( \varepsilon \), the above equation is in the form of standard singular perturbation problem. Setting \( \varepsilon \) to zero leads to the equation:

\[
0 = -\omega_r - i_r + u_r
\]

To obtain the reduced model as we discussed earlier we should substitute the root of this equation into the torque equation, thus we obtain the quasi-steady-state model:

\[
\frac{d\omega_r}{dt_r} = \omega_r - u_r
\]
Example 1.2  Consider the feedback control system of Figure 1.2:

![Figure 1.2: Actuator control with high-gain feedback](image)

The inner loop represents actuator control with high-gain feedback. The high-gain parameter is the integrator constant $k_1$. The plant is a single-input-single-output nth-order system represented by the state-space model $\{ A, B, C \}$. The nonlinearity $\Psi(.)$ is a first quadrant-third quadrant nonlinearity.

The state equation of the closed loop system is given by:

$$ \dot{x}_p = Ax_p + Bu_p $$

$$ \frac{1}{k_1} u_p = \Psi(u - u_p - k_2 C x_p) $$

Since we deal with the high gain feedback controller, the value of parameter $k_1$ is rather large and we can treat $\frac{1}{k_1}$ as $\varepsilon$ and therefore the above equations are in the form of singular perturbation problem. Since the nonlinearity is first quadrant-third quadrant nonlinearity, the only root of the second equation is given by:

$$ u - u_p - k_2 C x_p = 0 $$

So, the resulting reduced model is:

$$ \dot{x}_p = (A - B k_2 C) x_p + Bu $$
This result is also evident from the figure 1.2: setting $\varepsilon$ to zero is the same as assuming $k_f$ to have infinity value. This value causes the inner loop to behave like a constant gain equal to 1. Therefore the simplified model is given by

$$\dot{x}_p = Ax_p - Bk_f y + Bu$$

Which corresponds to the previous answer since $y = Cx_p$.

### 1.2 Singular Perturbed System Analysis:

In the previous section we have somehow learned how to model a typical physical problem into a singular perturbation model.

We have discussed that the singular perturbation model includes two categories of slow and fast variables. The reduced model actually ignores the fast transient response and is related to the slow response of the system. Recall that we have set $\varepsilon$ to zero to obtain the reduced model. But, having set the $\varepsilon$ to zero, will make the fast variable $z$ instantaneous, and there is no guarantee for the convergence of $z$ to its quasi-steady-state. This convergence must be held for the validity of our simplified model.

To skip some boring mathematical analysis let us consider this concept through an example:

**Example 1.3** Consider the singular perturbation problem for the DC motor of example 1.1:

\[
\begin{align*}
\dot{x} &= z \\
\dot{z} &= -x - z + u(t) \\
\varepsilon z &= -x - u(t)
\end{align*}
\]

Suppose $u(t) = t$ for $t > 0$ and we want to solve the state equation over the interval $[0, 1]$. 
The unique root of the fast state variable equation is:

\[ h(t, x) = -x + u(t) = -x + t. \]

To obtain the reduced model as we said earlier we should substitute the root \( h(t, x) \) into the slow state variable equation, Thus we have the reduced model as:

\[ \dot{x} = -x + t, \quad x(0) = \xi_0. \]

Since we want to analyze the convergence of the fast variable \( z \) to its quasi-steady-state, it is quite suitable to shift this quasi-steady-state (i.e. \( h(t, x) \)) to the origin. To do this task, we define new variable \( y \) such that \( y = z - h(t, x) \).

As a result the new system equations obtained as follows:

\[ \dot{x} = y + h(t, x) \]

\[ \varepsilon (y + h(t, x)) = -x - (y + h(t, x)) + u(t) \Rightarrow \varepsilon \dot{y} = -x - y - h(t, x) + u(t) - \varepsilon \frac{\partial h}{\partial t} - \varepsilon \frac{\partial h}{\partial x} z \]

With the initial state:

\[ x(0) = \xi_0 \]

\[ y(0) = \xi_0 + \eta_0 \]

As we discussed earlier due to the presence of fast and slow states, singular perturbation shows multi-time scale behaviour, therefore it seems appropriate that we analyze the slow and fast state variables in different time scale. Let us define the new time-scale \( \tau \) for the analysis of the fast transient dynamic. Note that we want to have a better view of what is happening for the fast transient, on the other hand all the events related to the fast transient occurs at a quite small interval after \( t_0 \). Therefore the time-axis should be stretched so that we can see these events much more clear. As a result we define:

\[ \tau = \frac{(t - t_0)}{\varepsilon} \]
Note that smaller $\varepsilon$ leads to the time-scale to be more stretched which is what we expect. As a result of this new time-scale for the fast state variables, we have:

$$
\dot{y} = \frac{dy}{dt} = \frac{dy}{d\tau} \times \frac{d\tau}{dt} = \frac{1}{\varepsilon} \frac{dy}{d\tau}
$$

Thus the fast state equation can be rewrite as:

$$
\frac{dy}{d\tau} = -x - y - h(t,x) + u(t) - \varepsilon \frac{\partial h}{\partial t} - \varepsilon \frac{\partial h}{\partial x} z
$$

(Note that the $\tau$ time-scale is only applied to the fast state variable $y$)

Keep in mind that variables $x$ and $t$ in the $\tau$ time-scale considered as slow variables since:

$$
t = t_0 + \varepsilon t \\
x(t) = x(t_0 + \varepsilon t)
$$

Setting $\varepsilon = 0$ in the fast state variable equation, we have:

$$
\left\{ \begin{align*}
\frac{dy}{d\tau} = -x - y - h(t,x) + u(t) \\
h(t,x) = -x + t \\
u(t) = t
\end{align*} \right\} \Rightarrow \frac{dy}{d\tau} = -y
$$

- The resulted equation is an important equation called “boundary layer model”.

- **In conclusion** of what we learned so far, we can say that the original singular perturbation problem is restated as two subsystems: the reduced model, and the boundary layer model. The first one is related to the slow state variables while the second one estimates the behaviour of the fast transients.
In our example we obtained these mentioned two models as:

\[
\begin{align*}
\frac{dx}{dt} &= -x + t \quad x(0) = \xi_0 \\
\frac{dy}{d\tau} &= -y \quad y(0) = \xi_0 + \eta_0
\end{align*}
\]

The reduced model response is:

\[ \tilde{x}(t) = (t - 1) + (1 + \xi_0)e^{-t} \]

The boundary layer system is globally exponentially stable and its response is:

\[ \tilde{y}(\tau) = (-\xi_0 + \eta_0)e^{-\tau}. \]

Therefore the approximate response of our original variable \( z \) would be equal to:

\[ \tilde{z}(t) = \tilde{y}(t) + h(t, x) \rightarrow \tilde{z}(t) = (\xi_0 + \eta_0)e^{-t} + 1 - (1 + \xi_0)e^{-t} \]

We can see from the above result that the approximation of the fast state variable \( \tilde{z} \), starts form the boundary layer and rapidly converges to the reduced model.

- Generally the reduced and boundary model can be obtained by the following equations:

\[
\begin{align*}
\frac{dx}{dt} &= f(t, x, h(t, x), 0) \quad (1.6) \\
\frac{dy}{d\tau} &= g(t, x, y + h(t, x), 0) \quad (1.7)
\end{align*}
\]
1.4 Geometric view of Singular Perturbation Problem:

In this section we want to give a geometric view of the singular perturbation problem including reduced and boundary model. First it is necessary to get familiar with the concept of *Integral Manifold*.

**Integral Manifold (Invariant Manifold):**

Consider the following differential equation:

\[ \dot{x} = N(t, X) \]  

(1.8)

where \( X, N \in \mathbb{R}^n \). The set \( M \subseteq \mathbb{R} \times \mathbb{R}^n \) is said to be an *integral manifold* if for \( \forall (t_0, X_0) \in M \), the solution \( (t, x(t)) \), \( X(t_0) = X_0 \), is in \( M \), for all \( t \in \mathbb{R} \).

If only for a finite interval of time, \( (t, x(t)) \in M \), then \( M \) is said to be a *local integral manifold*.

This definition implies that if the initial state start on \( M \) the trajectories of the system remains in \( M \) thereafter.

Now, let us illustrate the geometric view of our singular perturbation problem through a simple example:

**Example 1.4** Consider the singular perturbed system:

\[ \begin{align*}
\dot{x} &= -x + z \\
\dot{z} &= \varepsilon \tan^{-1}(1 - z - x)
\end{align*} \]

The second equation has the isolated root equal to: \( z = (1 - x) \). This equation represents a one-dimensional manifold in \( \mathbb{R}^2 \), this manifold is considered as the
slow manifold, since it refers to the reduced (slow) model. The corresponding slow model is obtained by: \( x = -2x + l \), which has an asymptotically stable equilibrium point \( P = (1/2,1/2) \). Therefore if the trajectories start on this slow manifold it would exponentially converges to point \( P \) and so this manifold also represents an integral manifold.

In order to have a overall view of the phase plane of the system I think it is helpful to look into the ratio \( \frac{dx}{dz} \). This ratio can be reconsidered as:

\[
\frac{dx}{dz} = \frac{dx}{dt} \cdot \frac{dt}{dz} = \varepsilon \times f(x,z) \quad g(x,z)
\]

(1.9)

To have an approximate view of the system phase plane, we set \( \varepsilon \) to zero. So from (1.9) we have \( \frac{dx}{dz} = 0 \), therefore the fast manifolds are straight lines parallel to the x-axis. Note that as \( \varepsilon \) increase from zero, these manifolds distort from these straight lines.

As a result an approximate phase portrait of the system can be shown by figure bellow:

![Figure 1.3: the approximate phase portrait of example 1.4](image-url)
It is useful to look at the boundary layer model in this problem: According to (1.7) the boundary layer model is:

\[ \frac{dy}{d\tau} = \tan^{-1}(-y) = -\tan^{-1}(y) \]

Form the above equation we can see that when the \( y > 0 \) \( (z > h(t,x)) \) then \( \frac{dy}{d\tau} < 0 \). Consequently the variable \( y \) decreases and the fast variable \( z \) converges to \( h(t,x) \) along the fast manifold. This is the case when in figure 1.5 the trajectories start from a point above the line \( z = -x + l \).

When \( y < 0 \) \( (z < h(t,x)) \) then \( \frac{dy}{d\tau} > 0 \). Consequently the variable \( y \) increases and the fast variable \( z \) converges to \( h(t,x) \) along the fast manifold. This is the case when in figure 1.3 the trajectories start from a point over the line \( z = -x + l \).

When the trajectory start on the line \( z = -x + l \) then \( y = 0 \) \( \frac{dy}{d\tau} = 0 \) and the trajectory does not distort from the line and converges to the equilibrium point according to the reduced model equation: \( x = -2x + l \).

1.5 Stability Analysis:

Here, we devote our attention to one of the theories in this field that has an extremely important concept. This theory verifies the robustness of exponential stability to the unmodeled fast dynamics. This theory also relates the stability of reduced model and boundary layer model to the whole (actual) system stability.

**Theorem:** Consider the singularly perturbed system:

\[ 
\begin{align*}
\dot{x} &= f(t, x, z, \varepsilon) \\
\varepsilon \dot{z} &= g(t, x, z, \varepsilon)
\end{align*}
\]
Assume that the following assumptions are satisfied for all 
$(t,x,\varepsilon) \in [0,\infty) \times B_\rho \times [0,\varepsilon_0]$
:

• $f(t,0,0,\varepsilon) = 0$ and $g(t,0,0,\varepsilon) = 0$
• The equation $0 = g(t,x,z,0)$ has an isolated root $z = h(t,x)$ such that $h(t,0) = 0$
• The functions $f$, $g$ and $h$ and their partial derivatives up to order 2 are bounded for $z - h(t,x) \in B_\rho$
• The origin of the reduced system

$\dot{x} = f(t,x,h(t,x),0)$

is exponentially stable.

• The origin of the boundary-layer system:

$$\frac{dy}{d\tau} = g(t,x,y + h(t,x),0)$$

is exponentially stable, uniformly in $(t,x)$.

Then, there exists $\varepsilon^* > 0$ such that for all $\varepsilon < \varepsilon^*$, the origin of the whole system is exponentially stable.

* * *

As we said earlier this theory indicates that if the boundary layer model equilibrium were stable, then our simplified model would be valid and the stability of this simplified model results in the stability of the actual system. Let us illustrate this fact with the following example:
Example 1.5 Consider the feedback stabilization of the system

\[
\dot{x} = f(t, x, Cz) \\
\epsilon \dot{z} = Az + Bu
\]

where \( f(t,0,0) = 0 \) and \( A \) is a Hurwitz matrix. The system equilibrium point is at origin, and we want to design a state feedback control law to stabilize the origin. The linear part of this model represents actuator dynamics which are, typically, much faster than the plant dynamics represented by the nonlinear equation.

- We know that to design the control law, we neglect the actuator dynamics and we consider only the slow fast variables \( x \).

Now, let us analyze the validity of our design. We want to see if ignoring the fast transient dynamic causes any problem?! and whether the designed feedback for the reduced model stabilize the actual system?!

We consider the control signal \( u \) in general form of \( \kappa(t,x) \).

To obtain the reduced model we have:

\[
0 = Az + B\kappa(t,x) \quad \Rightarrow h(t,x) = -A^{-1}B\kappa(t,x) \Rightarrow
\]

the reduced model:

\[
\dot{x} = f(t,x,-CA^{-1}B\kappa(t,x))
\]

Assuming \(-CA^{-1}B = I\) for the simplicity, We have:

\[
\dot{x} = f(t,x,\kappa(t,x))
\]

Keep in mind that the reduced model is the nominal model that we designed the state feedback according to. Therefore the stability of this reduced model is guaranteed.
The boundary layer model given according to (1.7) is:

\[
\frac{dy}{d\tau} = A(y - A^{-1}B\kappa(t,x)) + B\kappa(t,x) \Rightarrow \frac{dy}{d\tau} = Ay
\]

Since the matrix A is assumed to be Hurwitz, the equilibrium of this boundary layer is exponentially stable.

Assuming that \( f \) and \( \kappa \) are smoothed enough to satisfy the mathematical constraints of the mentioned stability theorem, we conclude that the origin of the actual closed-loop system is exponentially stable.

Therefore our state feedback design for the nominal system remains valid for the actual system.

CHAPTER 2:
Research Areas & Applications

2.1 Researches with the Analysis point of view:

Some researches deal with the stability properties of singularly perturbed system. The most common analysis is done through Lyapunov methods. The main idea is to consider two lower order systems: the reduced model and boundary-layer systems which already discussed in the previous chapter. Assuming that each of these two systems is asymptotically stable and has a Lyapunov function it can be shown that, for a sufficiently small perturbation parameter, asymptotic stability of the singularly perturbed system can be established by means of a Lyapunov function which is a weighted sum of the Lyapunov functions of the reduced and boundary-layer systems. You may find this Lyapunov based analysis in [2],[3].
Furthermore, in [3] this composite Lyapunov function is used to obtain estimates of the domain of attraction. The method given in [3] is however limited to a special case where the boundary-layer system is linear.

In [4] a general case in which the boundary-layer system is also nonlinear is studied and in addition to stability analysis, the quadratic-type composite Lyapunov function is used to obtain an upper bound on the perturbation parameter and also to estimate the domain of attraction. It is also shown that the choice of the weights of the composite Lyapunov function involves a trade off between obtaining a large estimate of the domain of attraction and a large upper bound on the perturbation parameter. 

In [10] an alternative stability analysis can be found: A kind of LFT form of the singular perturbed system has been proposed. Using linear fractional transformations, a singularly perturbed system is formulated into a standard $\mu$- interconnection framework. Also a set of new stability conditions for the system by defining the real structured singular value $\mu$ is derived. Note that this method is only limited to linear singular perturbed system.

- Those who are interested in Lur’se problem can find the circle criterion for the singular perturbed system in [11].

The stability analysis and stability properties of the Time Delayed Singularity Perturbed System is one of the interesting research subjects in the field of singular perturbed system. A small delay in the feedback loop of a singularly perturbed system may destabilize it; however, without the delay, it is stable for all small enough values of a singular perturbation parameter.

In [5] the stability of a linear time invariant singularly perturbed system with time-delay in slow states is discussed. An upper bound for perturbation parameter $\varepsilon^*$ is given, such that the stability of the full-order system can be inferred from the analysis
of the reduced-order model in separate time scales. It also provided a way to find the range for time delay such that the stability of the system is guaranteed for $0 \leq \varepsilon < \varepsilon^*$. 

- Generally two main approaches have been developed for the treatment of the effects of small delays: frequency domain techniques and direct analysis of characteristic equation. The stability of singularly perturbed systems with delays in the frequency domain can be found in [6] and [7]. However, the method of LMIs is more suitable for robust stability of systems with uncertainties and for other control problems. (see e.g. [8]).

In [9] sufficient and necessary conditions for preserving stability, for all small enough values of delay and $\varepsilon$, are given in two cases: in the case of delay proportional to $\varepsilon$ and in the case of independent delay and $\varepsilon$. Also the sufficient conditions are given in terms of an LMI for the second case.

### 2.2 Design Methods:

#### 2.2.1 Linear Singular Perturbed System:

- **Optimum Control Design:**

Consider the problem of finding the optimal state feedback gains solving the linear-quadratic regulator (LQR) problem for the following singularly perturbed linear, time-invariant system:

\[
\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u
\]

\[
\varepsilon \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u
\]

\[y = C_1x_1 + C_2x_2\]
The above system is said to be in standard form if $A_{22}$ is invertible. There are generally three representative options for solving this problem:

1. **The special structure of the above system may be ignored, and the full regulator algebraic Riccati equation (ARE) is solved.** This method leads to numerical difficulties. Further, we will have missed the chance to reduce the size and complexity of our controller which is a particularly important consideration for large systems. (see e.g. [11]).

2. **The second method is related to what we have illustrated in the previous chapter: transforming the actual system into two subsystems, the reduced model and the boundary layer model. Then the Regulator AREs are solved independently for these two subsystems to obtain slow and fast feedback gains, $K_s$ and $K_f$. Then the composite control $u = K_s x_s + K_f x_f$ an approximation of the true optimal value. (see e.g. [12]).**

This is a valuable result since two smaller computations are preferred to a single large one, and also the resulting compensator complexity is reduced. However, the near optimality of the solution (and even stability) is guaranteed asymptotically, only for sufficiently small $\varepsilon$.

- The extension of this design method into **Non-standard** case where $A_{22}$ is not invertible is discussed in [15].

- In [16] considering the reduced model and boundary layer model, the output feedback design is discussed in the discrete singular perturbed system.

3. **In this method first we construct the Hamiltonian form of the optimal closed-loop system, with new slow states consisting of $x_1$ and the slow co-states, and new fast states consisting of $x_2$ augmented by the fast co-states. Their results are**
motivated and enabled by the observation that the Hamiltonian system maintains its Hamiltonian structure under the exact decoupling transformation.

Manipulation of the decoupled pure-slow and pure-fast Hamiltonian systems results again in two AREs, which may be solved for the pure-slow and pure-fast state feedback gains. This approach differs from the second method in two important ways: The first is that the corresponding composite control in this approach is exactly the optimal control. The second is that the decoupled regulator AREs are non-symmetric. (see e.g. [13])

• The above methods were about the time-invariant system. In [14] a new method is proposed for the time-varying case.

• $H_\infty$ Control problem for singularly perturbed systems has been extensively studied in the 90s. (see e.g. [17],[18],[19]). In these articles it is showed that an $H_\infty$ sub-optimal controller for a singularly perturbed system can be obtained by a two-stage design procedure. First, design a $H_\infty$ sub-optimal controller for its fast subsystem, and then design an $H_\infty$ sub-optimal controller for the modified slow subsystem. It is obvious that the decomposability of the original singularly perturbed system must be assumed to guarantee the applicability of the two-stage procedure, thus their results apply only to standard singularly perturbed system.

In [20], a set of $\varepsilon$-independent conditions for the existence of an $H_\infty$ sub-optimal controller is derived for small $\varepsilon$ from the decomposition of the Riccati equations. This sub-optimal controller constructed form the total system is also singularly perturbed, and its fast (slow) part is an $H_\infty$ sub-optimal controller for the fast (slow) subsystems of the full system, respectively. Finally, a procedure for designing an independent $H_\infty$ suboptimal controller is proposed. The design method given in this paper is also applicable for non-standard singularly perturbed systems.
In [21], a robust sampled-data control method is proposed and state-feedback $H_\infty$ control problem for linear singularly perturbed systems with norm-bounded uncertainties is studied. Also Linear Matrix Inequalities (LMIs) criteria are derived for the system stability.

[22] deals with the design of robust controllers for linear time invariant uncertain singularly perturbed systems using Periodic Output Feedback. It is shown that a periodic output feedback controller designed from slow uncertain model will stabilize the actual full order system model for sufficiently small perturbation parameter (provided fast modes are asymptotically stable.)

### 2.2.2 Nonlinear Singular Perturbed System:

**Integral Manifold Based Approach:**

The technique of applying invariant manifold methods to singularly perturbed nonlinear differential equations was first recognized by Zadiraka in 1957. In his first paper [22] he showed the existence of a local integral manifold. In a later paper [23], he proved in 1965 the existence of a global integral manifold. A common feature in the invariant manifold method is that the integral manifolds were constructed by extrapolating the degenerate manifold (i.e. the manifold obtained when $\varepsilon = 0$). More techniques and results are also available for analyzing and constructing integral manifolds (see e.g. [24],[25]).

In [26] the basic elements of the integral manifold method in the context of control system design (namely, the existence of an integral manifold, its attractivity, and stability of the equilibrium) while the dynamics are restricted to the manifold, are studied. A controller is proposed with a composite control law that consists of a fast
component, as well as a slow component that was designed based on the integral manifold approach.

- In the previous chapter we have stated the definition of Integral Manifold. Now, let us study the concept of Integral Manifold Based Design in some details:

Note that we presented the exact definition of Integral Manifold in section 1.4. Recall that if M be an integral manifold, the definition of integral manifold implies that if the initial state start on $M$, the trajectories of the system remains in $M$ thereafter. Considering an autonomous system and denoting the fast variable by $w$ and the slow one by $x$, a schematic view of an integral manifold is showed in figure bellow:

![Figure 2.1: Integral Manifold of an autonomous system](image)

Fig 2.1 illustrates the three basic issues of integral manifolds that must be considered in the control design: First, the existence of an integral manifold $M$, needs to be established so that if the initial states start on $M$, the trajectory of the system remains on $M$ thereafter. Second, when restricted to the integral manifold $M$, the dynamics of the system should insure stability of the equilibrium.
Third, the integral manifold $M$ should be attractive so that if the initial conditions are off $M$ as shown in Fig 2.1, the solution trajectory asymptotically converges to $M$. In a control system design context, the challenge is therefore to devise an appropriate control law $u(x, w)$ that insures the existence of an attractive integral manifold, and furthermore, insures stability of the system when the dynamics are restricted to the integral manifold. Note that the fundamental advantage of an integral manifold approach to control system design is that once an attractive integral manifold is designed for a dynamical system, the stability problem of the original system reduces to a stability problem of a lower dimensional system on the manifold which is typically much easier to deal with.(Note that the fast and the slow manifold for the singular perturbed system have discussed in the first chapter).

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• Robust State and Output Feedback Design :

The problem of designing controllers for nonlinear systems with uncertain variables, is another interesting research subject.

In, [27] robust state feedback controllers were synthesized for nonlinear singularly perturbed systems with time-varying uncertain variables.

[28], studies the problem of synthesizing a robust output feedback controller for nonlinear singularly perturbed systems with uncertain variables, for which the fast subsystem is asymptotically stable and the slow subsystem is input/output linearizable and possesses input-to-state stable inverse dynamics. A dynamic controller is synthesized, through combination of a high-gain observer with a robust state feedback controller synthesized via Lyapunov's direct method that ensures boundedness of the state and achieves asymptotic attenuation of the effect of the uncertain variables on the output of the closed-loop system. The derived controller enforces the requested
objectives in the closed-loop system, for initial conditions, as long as the singular perturbation parameter is sufficiently small and the observer gain is sufficiently large.

- **Sliding Surface Design:**

In [29], *Sliding Surface Design for Singularly Perturbed Systems* is discussed and a composite control method through variable structure control design has been proposed.

In this regard, let us have some discussion about the relationship between the equilibrium manifold of a singularly perturbed to the sliding surface and also similar behaviour of fast time and slow time responses to the “reaching mode” and “sliding mode”:

As we discussed in first chapter the two-time-scale behaviour of a singularly perturbed system is characterized by a slow and a fast motion in the system dynamics. The slow motion, approximated by a reduced model, is usually related to a variable structure system as “sliding mode”. The fast transients, represented by a “boundary layer” correspond to the “reaching mode” before the state trajectory lies on the sliding surface. (to have some sense about these similarity, our discussion about fast and slow manifolds which was given in the first chapter is quite useful).

The composite control method (composition of fast and slow modes control) has a strong correlation with sliding surface design in that it seeks to determine an appropriate equilibrium manifold for the slow system to "stay" in. Although this manifold has an equivalent meaning to a sliding surface, there exist two important distinctions: First, the dimension of the equilibrium manifold is equal to the number of fast state variables; while that of a sliding surface is equal to the number of control variables. Secondly, the way to drive the system into the equilibrium manifold is by
properly designing the fast control part law, which renders an asymptotically stable boundary layer; while in variable structure design the sliding mode is attained directly by \( m \) (number of control variables) discontinuous controls. Lyapunov's approach can treated as a bridge to soften these differences; that is, to create a sliding motion that shares the same Lyapunov function as the feedback singularly perturbed system.

2.3 Applications:

Recently, much effort has focused on the stability criteria and control design of singularly perturbed systems. This is due not only to theoretical interests but also to the relevance of this topic in control engineering applications. Typical singularly perturbed systems include direct-drive robots, flexible joint robots, flexible space structures, armatured-controlled DC motors, high-gain control systems (recall example 1.2 in first chapter), flexible mechanical systems, tunnel diode circuits, control system of an airplane, control system of an inverted pendulum, and nonlinear time-invariant RLC networks.

As we discussed in section 2.2.2, in [26] the integral manifold design method is studied. The results are applied to the control problem of multi-body systems with rigid links and flexible joints in which the inverse of joint stiffness plays the role of the small parameter.

The proposed composite controller in [26] has the following properties:

(i) It enables the exact characterization and computation of an integral manifold
(ii) It makes the manifold exponentially attractive
(iii) It forces the dynamics of the reduced flexible system on the integral manifold to coincide with the dynamics of the corresponding rigid system (i.e. the one obtained by making stiffness very large) implying that any
control law that stabilizes the rigid system would stabilize the dynamics of the flexible system on the manifold.

• In [30] the proposed control (geometry approach) has been applied to the nonlinear automobile idle-speed control system. The parameterized co-ordinate transformation to transform the original nonlinear system into singularly perturbed model and the composite Lyapunov approach is then applied for output tracking.

• In [31] the singular perturbation approach is used to study the diagnosability of linear two-time scale systems. Based on a power series expansion of the slow manifold around $\varepsilon = 0$, higher order corrected models for both slow and fast subsystems are obtained. It is shown that if the original singularly perturbed system is fault diagnosable, a composite observer-based residual generator for the original system with actuator faults can be synthesized from the observers of the two separate subsystems.

• The economic dynamics of reservoir sedimentation management using the hydro suction-dredging sediment-removal system is analyzed in [32]. In this article, system dynamics depend on two interdependent hydraulic processes evolving at different rates. The accumulation of water impounded in the reservoir evolves on a ‘fast’ time scale, while the loss of water storage capacity to trapped sediments evolves on a ‘slow’ time scale. A multidimensional optimal control problem with singularly perturbed equations of motion is formulated. Singular perturbation method is applied to approximate a ‘slow’ manifold and reduce multi-dimensional solution space to the single-dimensional subspace confining long-term dynamics.

• For the application of singular perturbation in chemical processes, in [33] a singularly perturbed system of second-order differential equations is considered. This system describes the steady state of a chemical process that involves three species, two reactions (one of which is fast), and diffusion.
CHAPTER 3:
Case Study

Consider the following nonlinear singular perturbed system:

\[
\begin{align*}
\dot{x} &= \frac{x^2(1+t)}{z} \\
x(0) &= 1 \\
\epsilon z &= -[z + (1+t)x][z - (1+t)] \\
z(0) &= \eta_0
\end{align*}
\]  

(3.1) \quad (3.2)

This system is a bit more complicated than what we have studied in the first chapter, in a way that setting \( \epsilon = 0 \) in (3.2), we obtain three isolated roots. These three roots are:

(1) : \( z = -(1+t)x \)
(2) : \( z = (1+t)x \)
(3) : \( z = 0 \)

The corresponding boundary layer model to the first root according to (1.7) is given by:

\[
\frac{dy}{d\tau} = -y[y - (1+t)x]\left[y - (1+t)x - (1+t)\right]
\]

(3.3)

Using a Lyapunov candidate: \( V = \frac{1}{2}y^2 \),
\[
\frac{\partial v}{\partial y} = y y = -y^2 [y - (1 + t)x] [y - (1 + t)x - (1 + t)]
\]

\[
\downarrow
\]

\[
\frac{\partial v}{\partial y} = -y^2 [y - (1 + t)x]^2 + y^2 [y - (1 + t)x] 1 + t)
\]

In order to having a stable boundary layer model, the function \( \frac{\partial V}{\partial y} \), must be negative definite. It is seen that the first term in above equation is negative definite. To make the second term also negative definite, the condition: \( y < (1 + t)x \) must hold. (due to the existence of coefficient \( y^2 \) this constraints satisfy the condition \( \frac{\partial V}{\partial y} < -\epsilon\|y\|^2 \)).

Assuming \( y < (1 + t)x \), the corresponding reduced model for the first root is obtained as:

\[
\dot{x} = -x \quad x(0) = 1 \quad (3.4)
\]

For the second root the corresponding boundary layer model using (1.7) is given by:

\[
\frac{dy}{d\tau} = -y [y + (1 + t)x] [y + (1 + t)x + (1 + t)] \quad (3.5)
\]

Using again the Lyapunov candidate: \( V = \frac{1}{2} y^2 \), we have:

\[
\frac{\partial v}{\partial y} = y y = -y^2 [y + (1 + t)x] [y + (1 + t)x + (1 + t)]
\]
Similar to what we mentioned regard to the first root, here the constraint $y > -(1 + t)x$ must be hold for the stability of the boundary layer model (3.5).

Assuming this constraint is satisfied, we have the following reduced model:

\[
\dot{x} = x^2 \quad x(0) = 1 \quad (3.6)
\]

- Note that (3.6) has a solution $x = \frac{1}{(1-t)}$, therefore the simulation time must be sufficiently smaller than the escape time in 1 sec.

And finally for the third root ($z = 0$), the corresponding boundary layer model is:

\[
\frac{dy}{dt} = -y[y + (1 + t)x][y - (1 + t)] \quad (3.7)
\]

Let’s sketch the right-hand side function of (3.7):

According to figure 3.1, since the sign of $\dot{y}$ and $y$ are the same around the origin, the boundary layer model (3.7) is unstable. Therefore it does not give a valid reduced model.
Consequently, we focused our attention to the first and second root and the corresponding reduced and boundary models have been simulated. For both roots, the estimation of slow and fast variables according to reduced model and boundary layer model respectively is sketched in figure 3.2. The perturbation parameter $\varepsilon$, is set to 0.1

Recall that the constraint $y < (1+t)x$, were obtained regard to first root. We know $y = z - h(t, x) \rightarrow y = z - (- (1+t)x)$, thus the mentioned constraint can be written as: $z < 0$. As a result, the initial condition on $z$ must starts from a negative value. Similarly for the second root we have $z > 0$

Figure 3.2 shows the estimations in case of $\eta_0 = -0.3, \eta_0 = -2$ for the first root and $\eta_0 = +0.3, \eta_0 = +2$ for the second one:

![Figure 3.2](image.png)

*Figure 3.2: Two estimations of fast variable state $z$ due to different initial states, at $\varepsilon=0.1$*
• It was earlier discussed that the accuracy of singular perturbed analysis is directly dependent to the perturbed parameter $\varepsilon$. To study the effect of this parameter, in figure 3.3 the simulation results in case of $\varepsilon = 0.1$ and $\varepsilon = 0.5$ are compared:

![Simulation results for two different values of $\varepsilon$. Reduced solution (dashed), exact solution (solid).](image)

Two phenomenon are recognizable in figure above: First it is seen from figure above that the speed of transient response which is related to the fast variable state $z$, is different in two cases. This is obvious since we have:

$$\varepsilon \dot{z} = g(x,t,\varepsilon) \quad \Rightarrow \quad \dot{z} = \frac{1}{\varepsilon} g(x,t,\varepsilon)$$

Therefore, a smaller $\varepsilon$, result in a faster transient response.
Second, the figure 3.3 shows that state variables converges to their exact value faster in case of $\varepsilon = 0.1$. Also the estimation error in this case is fairly lower than the other case ($\varepsilon = 0.5$). This fact can be illustrated as when the value of $\varepsilon$ is smaller, as we said the rate of fast state response is higher, though we can say that when $\varepsilon$, accept a smaller value, the fast and slow variables are more decoupled. So, the estimation error is decreased and the convergence rate increases. Lets say the mentioned reason in another way: recall that the boundary layer model was achieved by setting $\varepsilon$ to zero; so, the boundary layer model and the corresponding reduced model are achieved on the assumption of $\varepsilon$ to be sufficiently close to zero. Therefore, it is logic that increasing the value of $\varepsilon$, will cause some imperfections.

**Conclusion:**

We have found the singular perturbed theory a powerful tool for design and analysis for systems show multiple-time scale behaviour.

We have reviewed important concept and theories involved in singular perturbation. We found integral manifolds as a useful definition both in analysis and design of singular perturbed system. In research area chapter, it was seen that the design methods for the singular perturbed system is completely an open issue. And in the last chapter we have learned that our estimations are fairly dependent to initial values and especially to the singular perturbation parameter.
References:


