V. Conclusion

It is shown how to add a stable parameter estimator reset algorithm to a nonlinear adaptive backstepping design. Particular attention is paid to transient performance, which is the main motivation of the reset mechanism. The simulation example shows the feasibility and benefits of the approach. The transient performance of a somewhat idealized wheel slip control system has been shown to be significantly improved by adding the reset algorithm.

References


An Antiwindup Approach to Enlarging Domain of Attraction for Linear Systems Subject to Actuator Saturation

Yong-Yan Cao, Zongli Lin, and David G. Ward

Abstract—This note considers linear systems subject to actuator saturation. An antiwindup technique is used to enlarge the domain of attraction of the closed-loop system under an a priori designed linear dynamic feedback law. The design of the antiwindup compensation gain is formulated and solved as an iterative optimization problem with LMI constraints. Numerical examples are used to demonstrate the effectiveness of the proposed design technique.

Index Terms—Actuator saturation, antiwindup, domain of attraction.

I. INTRODUCTION

The analysis and synthesis of controllers for dynamic systems subject to actuator saturation have been attracting increasingly more attention (see, for example, [2], [17], [12], and the references therein). There are mainly two approaches to dealing with actuator saturation. One approach is to take control constraints into account at the outset of control design. Considerable progress has been made in this area. For example, a nested feedback design technique was proposed in [18] to design nonlinear globally asymptotically stabilizing controller; a low-and-high

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gain method was presented in [17] to design linear semiglobally stabilizing controllers. In [5], the authors proposed the method of maximal output admissible sets to analyze the attraction region for linear systems in the presence of control and state constraints.

Another approach is to first ignore actuator saturation and design a linear controller that meets the performance specifications, and then design an antiwindup compensator to weaken the influence of saturation. This method was widely used in control engineering and there are many techniques for designing an antiwindup compensator. One of the first attempts to analyze windup control system stability was the application of the scalar Popov and circle criteria by Glattfelder et al. [6, 7] in the context of antireset windup control systems. The multivariable circle criterion was extended in [9] to design multivariable antireset windup control systems. Zheng et al. [19] used the off-axis circle criterion to establish stability of their antwindup scheme based on the internal model control approach. Attempts to use the scaled small gain theorem for the same purpose were reported in [4]. The other methods reported are describing function analysis [1], incremental gain analysis [16], invariant subspace technique [14] and multiplier theory [11]. A unified framework for the study of antwindup design was presented in [10], which is known to be equivalent to the observer type approach [15].

In this note, we will explore possibility of enlarging domain of attraction for the system with actuator saturation by antiwindup compensation. We will first obtain an estimate of the domain of attraction under a given antiwindup compensation. This estimate is then maximized over the choice of the antiwindup compensation gain. It is known that the estimates of the domain of attraction made by small gain theorem, Popov criterion or circle criterion are sometimes conservative. In [8], a less conservative analysis approach is proposed to analyze the stability and the domain of attraction for the systems with actuator saturation. The idea is to formulate the analysis problem into a constrained optimization problem with constraints given by a set of linear matrix inequalities (LMI’s). In this note, we will utilize the method of [8] to arrive at an estimate of the domain of attraction under a given antiwindup compensation. An iterative LMI approach will be proposed to design the antiwindup compensation gain which optimizes this estimate of the domain of attraction.

The note is organized as follows. Section II gives some preliminary results and states more precisely our problem formulation. Stability and domain of attraction of the closed-loop system with actuator saturation and antiwindup compensation are analyzed in Section III. An iterative LMI algorithm is proposed to design antiwindup compensation gain to enlarge the domain of attraction. Numerical examples illustrating our design procedure and its effectiveness are given in Section IV. The note is concluded in Section V.

II. PRELIMINARIES AND PROBLEM STATEMENT

Let us consider the linear system with actuator saturation

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) the control input, \( y(t) \in \mathbb{R}^r \) the measured output, and \( A, B \) and \( C \) are real-valued matrices of appropriate dimensions. The function \( \sigma : \mathbb{R}^m \rightarrow \mathbb{R}^n \) is the standard vector-valued saturation function defined as follows:

\[ \sigma(u) = [\sigma(u_1) \ \sigma(u_2) \ \cdots \ \sigma(u_m)]^T \]

where \( \sigma(u_i) = \text{sign}(u_i) \min\{1, |u_i|\} \). Here we have slightly abused the notation by using \( \sigma \) to denote both the scalar valued and the vector valued saturation functions. Also note that it is without loss of generality to assume unity saturation level. The nonunity saturation level can be absorbed into the \( B \) matrix (see the example in Section 4).

We will assume that a linear dynamic compensator of the form

\[ \dot{x}_c = A_c x_c + B_c y, \quad x_c(0) = 0 \] (3)
\[ u = C_c x_c + D_c y \] (4)

has been designed that stabilizes the system (1)–(2) in the absence of actuator saturation and meets performance specifications.

Actuator saturation can lead to deterioration in the performance of the control system, and may also lead to instability. When the actuator saturates, the performance of the closed-loop system designed without considering the saturation may seriously deteriorate. In general, the performance degradation is caused by the fact that the states of the controller achieve different values from those in the absence of actuator saturation [9]. This can result in improper control signals, and, consequently, deterioration in closed-loop performance; and may induce a limit cycle or an unstable output response. A well-known example of performance degradation (e.g., large overshoot and large settling time) occurs when a linear compensator with integrators, say a PID compensator, is used in a closed-loop system and the reset-windup phenomenon appears. During the time when the actuators saturate, the error is continuously integrated even though the controls are not what they should be, and hence, the states of the compensator attain values that lead to larger controls than the saturation limit. This is known as the windup phenomenon. A common approach in practice is to perform a linear control design, then to add extra feedback compensation at the stage of control implementation, using the difference between the controller output signal and the saturated control signal. As this form of compensation aims to reduce the undesirable effects of windup, it is referred to as antiwindup.

A typical antiwindup compensator involves adding a “correction” term of the form \( E_c(\sigma(u) - u) \). The modified compensator has the form

\[ \dot{x}_c = A_c x_c + B_c y + E_c(\sigma(u) - u), \quad x_c(0) = 0 \]
\[ u = C_c x_c + D_c y \]

where \( x_c \in \mathbb{R}^n \). Obviously, with such a correction term, the compensator (5) would continue to behave like the dynamic controller (3) in the absence of saturation, i.e., \( \sigma(u) = u \), and the compensation in (5)–(6) has been made in a sufficiently smooth manner so that existence and uniqueness of solutions for the closed-loop system are guaranteed.

Under the compensated controller, the closed-loop system can be written as,

\[ \dot{x} = \tilde{A} \dot{x} + \tilde{B}(\sigma(u) - u) \]
\[ u = F \dot{x} \]

where

\[ \tilde{A} = \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} A + BD_c C & BC_c \\ B_c C & A_c \end{bmatrix} \],
\[ \tilde{F} = \begin{bmatrix} D_c C & C_c \end{bmatrix}. \]

With the feedback control law (8), the system (7) can be rewritten as

\[ \dot{x} = (\tilde{A} - \tilde{B} \tilde{F}) \dot{x} + \tilde{B} \sigma(F \dot{x}). \] (9)
In this note, we will first obtain an estimate of the domain of attraction of the system (9) under a given compensation gain matrix $E_c$. This estimate is then optimized over the choice of $E_c$.

Let $f_i$ stand for the $i$-th row of the matrix $F$. We define the symmetric polyhedron

$$
\mathcal{L}(F) = \{ \dot{x} \in \mathbb{R}^{n+n_c} : f_i \dot{x} \leq 1, i = 1, \ldots, m \}.
$$

For $\dot{x}(0) = \tilde{x}_0 \in \mathbb{R}^{n+n_c}$, denote the state trajectory of the system (9) as $\varphi(t, \tilde{x}_0)$. Then the domain of attraction of the origin is $\mathcal{S} := \{ \tilde{x}_0 \in \mathbb{R}^{n+n_c} : \lim_{t \to +\infty} \varphi(t, \tilde{x}_0) = 0 \}$. A set is said to be invariant if all the trajectories starting from it will remain in it [3].

Let $P \in \mathbb{R}^{(n+n_c) \times (n+n_c)}$ be a positive-definite matrix. Denote

$$
\Omega(P, \rho) = \{ \dot{x} \in \mathbb{R}^{n+n_c} : \dot{x}^T P \dot{x} \leq \rho \}.
$$

Let $V(\dot{x}) = \dot{x}^T P \dot{x}$. The set $\Omega(P, \rho)$ is said to be contractively invariant if $V(\dot{x}) = 2\dot{x}^T P(\dot{A} - BF + \dot{B}M(\nu, F, H)) \leq 0$ for all $\dot{x} \in \Omega(P, \rho) \setminus \{0\}$. Clearly, if $\Omega(P, \rho)$ is contractively invariant, then it is inside the domain of attraction.

For any two matrices $F, \dot{H} \in \mathbb{R}^{m \times (n+n_c)}$ and a vector $\nu \in \mathbb{R}^m$, denote

$$
M(\nu, F, \dot{H}) := \text{diag}[\nu_1, \nu_2, \ldots, \nu_m]F + (I - \text{diag}[\nu_1, \nu_2, \ldots, \nu_m])\dot{H}.
$$

Let $\mathcal{V} = \{ \nu \in \mathbb{R}^m : \nu_i = 1 \text{ or } 0 \}$. There are $2^m$ elements in $\mathcal{V}$. We will use a $\nu \in \mathcal{V}$ to choose from the rows of $F$ and $\dot{H}$ to form a new matrix $M(\nu, F, \dot{H})$: if $\nu_i = 1$, then the $i$-th row of $M(\nu, F, \dot{H})$ is $f_i$, the $i$-th row of $F$, and if $\nu_i = 0$, then the $i$-th row of $M(\nu, F, \dot{H})$ is $\dot{h}_i$, the $i$-th row of $\dot{H}$.

**Lemma 1** [8]: Given an ellipsoid $\Omega(P, \rho)$, if there exists an $\hat{H} \in \mathbb{R}^{m \times (n+n_c)}$ such that

$$
(\dot{A} - BF + \dot{B}M(\nu, F, \dot{H}))^T P + P(\dot{A} - BF + \dot{B}M(\nu, F, H)) < 0, \quad \forall \nu \in \mathcal{V}
$$

and $\Omega(P, \rho) \subseteq \mathcal{L}(\hat{H})$, i.e., $|\dot{h}_i| \leq 1$ for all $\dot{x} \in \Omega(P, \rho), i = 1, 2, \ldots, m$, then $\Omega(P, \rho)$ is a contractively invariant set of the system (9).

As shown in [8], Condition (10) is less conservative than several existing conditions for set invariance of the systems with actuator saturation. With all the ellipsoids satisfying the set invariance conditions of Lemma 1, we may choose the “largest” one to obtain the least conservative estimate of the domain of attraction by the method of [8]. We will measure the largeness of the ellipsoids with respect to a shape reference set. Let $\lambda_{\mathcal{S}} \in \mathbb{R}^{n+n_c}$ be a prescribed bounded convex set containing origin. For a set $\mathcal{S} \subseteq \mathbb{R}^{n+n_c}$ which contains origin, define

$$
\alpha_{\mathcal{S}}(\mathcal{S}) := \sup \{ \alpha > 0 \mid \alpha \lambda_{\mathcal{S}} \subseteq \mathcal{S} \}.
$$

Obviously, if $\alpha_{\mathcal{S}}(\mathcal{S}) \geq 1$, then $\lambda_{\mathcal{S}} \subseteq \mathcal{S}$. Two typical types of $\lambda_{\mathcal{S}}$ are the ellipsoid

$$
\lambda_{\mathcal{S}} = \{ \dot{x} \in \mathbb{R}^{n+n_c} : \dot{x}^T R \dot{x} \leq 1 \} , \quad R > 0
$$

and the polyhedron

$$
\lambda_{\mathcal{S}} = \text{conv} \{ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l \}
$$

where $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l$ are a priori given points in $\mathbb{R}^{n+n_c}$.

**Lemma 1** gives a condition for an ellipsoid to be inside the domain of attraction. With the above reference sets, we can choose from all the $\Omega(P, \rho)$’s that satisfy the condition such that the quantity $\alpha_{\mathcal{S}}(\Omega(P, \rho))$ is maximized. This problem can be formulated as

$$
\begin{align*}
\max_{P > 0, \rho, \mathcal{S}} & \quad \alpha_{\mathcal{S}}(\Omega(P, \rho)), \\
\text{s.t.} & \quad a) \alpha_{\mathcal{S}}(\lambda_{\mathcal{S}}) \subseteq \Omega(P, \rho), \\
& \quad b) (\dot{A} - BF + \dot{B}M(\nu, F, H))^T P + P(\dot{A} - BF + \dot{B}M(\nu, F, H)) < 0, \quad \forall \nu \in \mathcal{V}, \\
& \quad c) \|\dot{h}_i\| \leq 1, \quad \forall \dot{x} \in \Omega(P, \rho), \quad i = 1, 2, \ldots, m. \quad (11)
\end{align*}
$$

With the given compensation gain $E_c$, the above optimization constraints are equivalent to some LMI’s. If $\lambda_{\mathcal{S}}$ is a polyhedron, then by Schur complement, Condition $a$) is equivalent to

$$
\alpha_{\mathcal{S}}^2 \frac{\dot{x}^T}{\rho} \leq 1 \iff \left[ \frac{\dot{x}^T}{\rho} (\frac{P}{\rho})^{-1} \right] \geq 0, \quad i = 1, 2, \ldots, l.
$$

In the following we will denote $Q = (\frac{P}{\rho})^{-1}$. If $\lambda_{\mathcal{S}}$ is an ellipsoid $\{ \dot{x} \in \mathbb{R}^{n+n_c} : \dot{x}^T R \dot{x} \leq 1 \}$, then $a)$ is equivalent to

$$
\alpha_{\mathcal{S}}^2 \frac{\dot{x}^T}{\rho} \leq 1 \iff R \leq \alpha_{\mathcal{S}}^2 Q.
$$

Condition $b$) is equivalent to

$$
Q(\dot{A} - BF + \dot{B}M(\nu, F, H))^T + (\dot{A} - BF + \dot{B}M(\nu, F, H))Q \leq 0, \quad \forall \nu \in \mathcal{V}.
$$

Condition $c$) is equivalent to

$$
\alpha_{\mathcal{S}}^2 \frac{\dot{x}^T}{\rho} \leq 1 \iff \left[ \frac{1}{Q} h_i Q \right] \geq 0, \quad i = 1, 2, \ldots, m.
$$

Let $\gamma = \alpha_{\mathcal{S}}^2$, and $G = HQ$. Also let the $i$-th row of $G$ be $g_i$, i.e., $g_i = h_i Q$. Note that $M(\nu, F, H)Q = M(\nu, FQ, HG)$, so $M(\nu, FQ, G)$. If $\lambda_{\mathcal{S}}$ is a polyhedron, then from the above derivation the optimization problem (11) can be reduced to the following one with LMI constraints:

$$
\begin{align*}
\min_{Q > 0, \nu, \mathcal{S}} & \quad \gamma, \\
\text{s.t.} & \quad a) \gamma \geq 0, \quad i = 1, 2, \ldots, l, \\
& \quad b) Q(\dot{A} - BF)^T + (\dot{A} - BF)Q + \dot{B}M(\nu, FQ, G) + M(\nu, FQ, G)\dot{B}^T < 0, \quad \forall \nu \in \mathcal{V}, \\
& \quad c) \left[ \frac{1}{Q} g_i \right] \geq 0, \quad i = 1, 2, \ldots, m. \quad (12)
\end{align*}
$$

On the other hand, if $\lambda_{\mathcal{S}}$ is an ellipsoid, Condition $a$) of (12) should be changed to

$$
\alpha_{\mathcal{S}}^2 \frac{\dot{x}^T}{\rho} \leq \gamma Q. \quad (13)
$$

**Remark 1**: This optimization problem leads to an estimate of the domain of attraction of the closed-loop system (7)–(8). As is proven in [8], this estimate is less conservative than several estimates found in the literature. Our objective in this note is to use $E_c$ as a free design parameter to enlarge the domain of attraction. Note that Conditions $a$ and $c$ can also be formulated in terms of second-order cones [13].
III. Antwindup Compensation Gain Design

In this section, we will present an iterative design approach to obtain the antwindup compensation gain such that the domain of the attraction may be as large as possible.

Note that Condition b) in optimization problem (12) is not linear in $E_c, Q$ and $G$. We also note that the nonlinear matrix inequality (10) cannot be reduced to an LMI in $E_c, P$ and $H$ simultaneously. This implies that we cannot obtain the antwindup compensation gain by directly solving an LMI optimization problem. In the following we will present an iterative LMI approach to design the antwindup compensation gain $E_c$. Denote

$$ P = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}, \quad \hat{B}_0 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{P}_2 = \begin{bmatrix} P_{12} \\ P_2 \end{bmatrix} \tag{14} $$

where $P_1 \in \mathbb{R}^{n \times n}, P_2 \in \mathbb{R}^{n \times n}$ and $P_{12} \in \mathbb{R}^{n \times n}$. Then $P\hat{B}_0 = \hat{P}_2 E_c$, and the nonlinear matrix inequality (10) can be rewritten as

$$ \hat{A}^T P + P \hat{A} + (M(\nu, F, H) - F)^T (P\hat{B}_0 + \hat{P}_2 E_c) + (P\hat{B}_0 + \hat{P}_2 E_c) (M(\nu, F, H) - F) < 0 $$

which is an LMI in $P_1$ and $E_c$. This implies that, with fixed $P_2$ and $H$ and $\tilde{G}$, we can determine an $E_c$ such that $P_1$ is as “small” as possible, i.e., making the region $\{ \tilde{E} \in \mathbb{R}^n : \tilde{x}^T P \tilde{x} \leq 1 \}$ as large as possible. Hence, we can present the following iterative algorithm.

Algorithm (Iterative Algorithm for Determining Antwindup Compensation Gain):

Step 1) For a given reference set $\lambda_R$ and $E_c = 0$, Solve the optimization problem (12). Denote the solution as $\gamma_0, Q_0$ and $G_0$. Set $\lambda_R = \gamma_0^{-1/2} \lambda_R$.

Step 2) Set $E_c$ with an initial value, $i = 1$ and $\gamma_{\text{opt}} = 1$.

Step 3) Solve the optimization problem (12) for $\gamma, Q$ and $G$. Denote the solution as $\gamma_i, Q_i$ and $G_i$ respectively.

Step 4) Let $\gamma_{\text{opt}} = \gamma_i \gamma_{\text{opt}}$, $\lambda_R = \gamma_i^{-1/2} \lambda_R, P = Q^{-1}, H = GQ^{-1}$.

Step 5) IF $|\gamma_i - 1| < \delta$, a pre-determined tolerance, GOTO Step 7), ELSE GOTO Step 6).

Step 6) Solve the following LMI optimization problem:

$$ \min_{P_1 \geq 0, E_c} \gamma_i $$

s.t.

a) $P \leq \gamma_i R$

b) $\hat{A}^T P + P \hat{A} + (M(\nu, F, H) - F)^T (P\hat{B}_0 + \hat{P}_2 E_c) + (P\hat{B}_0 + \hat{P}_2 E_c) (M(\nu, F, H) - F) < 0$,

c) $P \succeq \hat{b}_j \hat{b}_j^T, \ j = 1, 2, \ldots, m$ \tag{15}

where $P_1$ and $\hat{P}_2$ are defined in (14). Set the solution as $E_c$ and $i = i + 1$, then GOTO Step 3).

Step 7) IF $\gamma_{\text{opt}} \leq 1$, then, $\alpha_{\text{opt}} = (\gamma_{\text{opt}})^{-1/2}$ and $E_c$ is a feasible solution and STOP, ELSE set $E_c$ with another initial value and GOTO Step 2).

Remark 2: In Step 6), if $\lambda_R$ is a polyhedron, Condition a) of (15) should be changed to

$$ \lambda - \hat{x}_i^T P \hat{x}_i \succeq 0, \ i = 1, 2, \ldots, l. $$

It is easy to find that the two optimization problems in Steps 3) and 6) always have solutions with the resulting $\gamma_i \leq 1$ for $i \geq 2$. If in Step 2), the initial condition of $E_c$ is set to 0, then $\gamma_i \leq 1$ for all $i \geq 1$.

Remark 3: As is usual in nonlinear optimization problems, the optimization result depends on the given initial condition of $E_c$. If we set 0 as the initial value of $E_c$, we can always obtain a better domain of attraction than the system without antwindup compensation although it may not result in the best $\alpha_{\text{opt}}$. In general, we can select several typical initial values of $E_c$ and choose the largest $\alpha_{\text{opt}}$ and the corresponding $E_c$ as the “optimal” solution.

Remark 4: In practical control design, it is always desirable to design an antwindup compensation gain $E_c$ with limited value. This implies that, with $E_c = [\varphi_{ij}]_{i, j = 1, 2, \ldots, m}$, the compensation gain may be constrained element-by-element in the following form:

$$ \varphi_{ij} \leq \psi_{ij} \leq \tilde{\psi}_{ij}, \ i = 1, 2, \ldots, n_c, \ j = 1, 2, \ldots, m. \tag{16} $$

These constraints are linear and hence can also be incorporated naturally in the optimization process. These extra constraints can also prevent the numerical stiffness.

IV. NUMERICAL EXAMPLES

Consider the following benchmark example [4]:

$$ x_1 = -0.1 x_1 + 0.5 \text{sat}(u_1) + 0.4 \text{sat}(u_2) $$

$$ x_2 = -0.1 x_2 + 0.4 \text{sat}(u_1) + 0.3 \text{sat}(u_2) $$

where $u_1$ and $u_2$ are limited to $[-3, 3]$ and $[-10, 10]$, respectively. At time $t = 0$, the outputs $x_1$ and $x_2$ are subject to pulse set-point changes of duration 5 s and magnitudes 0.6 and 0.4, respectively. The following PI controller was considered in [4]:

$$ \dot{x}_1 = y_{sp1} - x_1 + e_1 [\text{sat}(u_1) - u_1] + e_2 [\text{sat}(u_2) - u_2] $$

$$ \dot{x}_2 = y_{sp2} - x_2 + e_2 [\text{sat}(u_1) - u_1] + e_2 [\text{sat}(u_2) - u_2] $$

$$ u_1 = 10(y_{sp1} - x_1) + x_1 $$

$$ u_2 = -10(y_{sp2} - x_2) + x_2 $$

where $y_{sp1}$ and $y_{sp2}$ refer to the set-points for the outputs. In the absence of saturation, this PI controller places the closed-loop poles at $\{-1, -1, -0.1, -0.1\}$. In [4], a new design approach is presented and the comparison results are given by simulation.

To apply our result, we set

$$ A = \begin{bmatrix} -1.0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.4 & 3 \\ 0.4 & 0.3 & 0 \end{bmatrix} $$

$$ C = \begin{bmatrix} 1.5 & 4 \\ 1.2 & 3 \end{bmatrix} $$

$$ A_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} -1 \\ 0 \end{bmatrix} $$

$$ C_c = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 10 \end{bmatrix}, \quad D_c = \begin{bmatrix} 0.3333 \\ 0 \\ 0 \\ -0.1 \end{bmatrix} $$

Based on the possible pulse setpoint changes, we let $\lambda_R = [x^T(0) \ x^T(0)]^T$, with $x(0) = [0.6 \ 0.4]^T$ and $x(0) = 0$. Also let $E_{c0} = 0$. Applying Lemma 1, we obtain $\alpha = 0.7844$. This implies that the estimate of the domain of attraction does not include the above given initial point. Using the above data as initial conditions in the iterative algorithm of the previous section and setting the gain constraint $\psi_{ij} = -\tilde{\psi}_{ij} = 100$, we obtain $\alpha = 2579.8$, and $E_{c0} = [13.8414 \ -88.3918 \ 13.9838 \ -99.9964]$. If we set the initial condition to be $E_{c0} = [10 \ 10 \ 10 \ 10]$, we obtain $\alpha = 2589.6$, and $E_{c0} = [14.2642 \ -84.9037 \ 14.8691 \ -99.9980]$. The ellipsoid corresponding to the latter case with $x_c = 0$ is shown in Fig. 1. The dashed curves are the state trajectories with the state initial conditions on the boundary of this ellipsoid and the controller initial state $x_c(0) = 0$. Obviously, all trajectories converge to the origin.
With the antiwindup compensation gain $E_c$ given in [4]

$$E_c = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 0.3 & 0 \\ 0 & -1 \end{bmatrix}$$

and by solving the LMI optimization problem (12), we obtain a much smaller ellipsoid with $\alpha' = 0.7896$. In fact, we can easily find some points on the ellipsoid shown in Fig. 1, but outside the ellipsoid corresponding to the above $E_c$, from which the trajectories of the closed-loop system do not converge to the origin. Fig. 2 shows a state response under the compensation gain given in [4]. The initial condition is $\dot{x}(0) = [-0.5483 - 0.4113 0 0]^T$. The trajectories converge to a new equilibrium point rather than the origin. This shows that the compensation gain given in [4] results in a rather small domain of attraction. Fig. 3 shows the state responses under the pulse setpoint changes mentioned above. These responses are very similar to those under the antiwindup compensation of [14].

Our design method also leads to an estimate of the stability region even for open-loop unstable plants. To demonstrate its effectiveness with unstable plants, let us change the first equation of the system to

$$\dot{x}_1 = 0.1x_1 - 0.1x_2 + 0.5\text{sat}(u_1) + 0.4\text{sat}(u_2).$$

The resulting plant is unstable with two poles at $0.1236 \pm j0.3236$. It can be verified that the closed-loop system without input saturation under the above PI controller is still stable. In the absence of $E_c$ compensation term, applying Lemma 1, we obtain $\alpha = 0.7427$. Now, with $E_c^0 = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}$, by the iterative algorithm, we obtain $\alpha = 61.2973$, and $E_c = \begin{bmatrix} 68.4408 & -99.5146 \\ 60.5044 & -99.9842 \end{bmatrix}$. The resulting ellipsoid with $x_c = 0$ is shown in Fig. 4. The dashed curves are the state trajectories with the state initial conditions on the boundary of this ellipsoid and the controller initial state $x_c(0) = 0$. The trajectories all converge to the origin. Plotted in Fig. 4 in dash-dotted curve is a diverging trajectory of the closed-loop system without the $E_c$ compensation term. The starting point is $[0.5188 \ 0.3970 \ 0 \ 0]^T$, which is chosen slightly outside the obtained ellipsoid corresponding to $E_c = 0$. Since it starts from close to the origin, it is clear that without the $E_c$ compensation term, the domain of attraction is very small.

V. CONCLUSION

In this note, we have proposed to use the antiwindup design technique to enlarge the stability region of a predesigned closed-loop system under actuator saturation. An iterative algorithm is presented to design the antiwindup compensation gain by the linear matrix inequality technique. Numerical examples show the effectiveness of the proposed method.
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