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State-space formulae for all stabilizing controllers that satisfy an H_∞-norm bound and relations to risk sensitivity

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Abstract: Given a linear system, all stabilizing controllers such that a specified closed-loop transfer function has H_∞ norm less than a given scalar, are parametrized. This characterization involves the solution to two algebraic Riccati equations, each with the same order as the system, and further gives feasible controllers also with this order. The relationship to the risk-sensitive LQG stochastic control problem is established, giving an equivalence between robust and stochastic control.

Keywords: H_∞ control, State space control, Riccati equation, Sensitivity, Robustness.

1. Introduction and problem formulation

Since the work of Zames [20], there has been much interest in the design of feedback controllers for linear systems that minimize the H_∞ norm of a specified closed-loop transfer function (see Francis [8] and the references therein). In particular, let a linear system be described by the state equation

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \tag{1}$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t), \tag{2}$$

$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t). \tag{3}$$

The signals are as follows: $w(t) \in \mathbb{R}^{m_1}$ is the disturbance vector; $u(t) \in \mathbb{R}^{m_2}$ is the control input vector; $z(t) \in \mathbb{R}^{p_1}$ is the error vector; $y(t) \in \mathbb{R}^{p_2}$ is the observation vector; and $x(t) \in \mathbb{R}^n$ is the state vector. It is also assumed that any frequency-dependent weights are included in this model.

The transfer functions will be denoted as

$$\begin{aligned} P(s) &:= \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \\ &= \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI - A)^{-1} [B_1 \quad B_2] \\ &\overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \\ &=: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \end{aligned} \tag{4}$$

For a linear controller with transfer function $K(s)$ connected from y to u, the closed-loop transfer function from w to z will be denoted

$$\mathcal{F}_\ell(P, K) := P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}. \tag{5}$$

The 'H_∞ control problem' is then to choose a controller, $K(s)$, that makes the closed-loop system internally stable (see Francis [8]) and minimizes $\| \mathcal{F}_\ell(P, K) \|_\infty$, where

$$\| E \|_\infty = \sup_\omega \bar{\sigma}(E(j\omega)),$$

and $\bar{\sigma}(\cdot)$ denotes the maximum singular value.

We will in fact be considering the closely-related problem of finding all stabilizing K such that

$$\| \mathcal{F}_\ell(P, K) \|_\infty < \gamma \tag{6}$$

for some prespecified $\gamma \in \mathbb{R}$. A complete solution to this problem is available [4,8] via the parametrization of all stabilizing controllers and a solution to the resulting model-matching problem [3]. The latter problem could be solved via a sequence of spectral factorization problems of increasing degree that reduces the problem to a Nehari exten-

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sion problem which can be solved via the state-space method in [10]. However, this complete procedure is both theoretically and computationally very involved; although simplifications seemed likely since Limebeer and Halikias [14] had shown that there were controllers of degree $\leqslant n$ at least in the case when $m_1 = p_2$ or $m_2 = p_1$.

It is the purpose of the present note to give a state-space parametrization of all controllers that satisfy (6); this solution will only involve two algebraic Riccati equations, each of degree n. (Complete derivations of these results will be appearing elsewhere, see [5] for a simplified case and for some relations to other approaches.)

2. Characterizing all solutions

This section will give a state-space characterization of all stabilizing controllers $K(s)$ such that $\|\mathcal{F}_\ell(P, K)\|_\infty < \gamma$. We will make the following assumptions that are also typically made in the corresponding Linear/Quadratic/Gaussian (LQG) problems.

A1. (A, B_2, C_2) is stabilizable and detectable. (This is required for the existence of a stabilizing K.)

A2. rank $D_{12} = m_2$, rank $D_{21} = p_2$. (This is sufficient to ensure that the controllers are proper, but there are sensible problems when it is violated.)

A3. A scaling of u and y, together with a unitary transformation of w and z, enables us to assume without loss of generality that (by A2)

$$D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$$

$$\begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix} \begin{matrix} \updownarrow p_1 - m_2 \\ \updownarrow m_2 \end{matrix}$$
$$\quad \overset{\leftrightarrow}{m_1 - p_2} \quad \overset{\leftrightarrow}{p_2}$$

A4. $D_{22} = 0$ (this will be removed later).

The final assumptions ensure that the solution to the corresponding LQG problem is closed-loop asymptotically stable, and is also convenient for the present problem.

A5. $\operatorname{rank}\begin{pmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m_2 \quad \forall \omega \in \mathbb{R}.$

A6. $\operatorname{rank}\begin{pmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{pmatrix} = n + p_2 \quad \forall \omega \in \mathbb{R}.$

(Note that A5 implies rank $P_{12}(j\omega) = m_2$ $\forall \omega$, but if this realization of P_{12} is not minimal, then the converse is not true. Similarly for A6.)

The solution to an algebraic Riccati equation (ARE) will be denoted via its Hamiltonian matrix, as

$$X = \operatorname{Ric}\begin{pmatrix} A & -P \\ -Q & -A^* \end{pmatrix}, \quad P = P^*, \ Q = Q^*,$$

where this implies that $X = X^*$ and

$$\begin{pmatrix} A & -P \\ -Q & -A^* \end{pmatrix}\begin{pmatrix} I \\ X \end{pmatrix} = \begin{pmatrix} I \\ X \end{pmatrix}(A - PX),$$

$\operatorname{Re} \lambda_i(A - PX) < 0.$

Now define

$$R = D_{1\cdot}^* D_{1\cdot} - \begin{pmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{7}$$

where

$$D_{1\cdot} = [D_{11} \quad D_{12}],$$

and

$$\tilde{R} = D_{\cdot 1} D_{\cdot 1}^* - \begin{pmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

where

$$D_{\cdot 1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}.$$

Now define X_∞ and Y_∞ as solutions to the following ARE's (assuming solutions exist):

$$X_\infty = \operatorname{Ric}\left\{\begin{pmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{pmatrix} - \begin{pmatrix} B \\ -C_1^* D_{1\cdot} \end{pmatrix} R^{-1} [D_{1\cdot}^* C_1 \quad B^*]\right\}, \tag{9}$$

$$Y_\infty = \operatorname{Ric}\left\{\begin{pmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{pmatrix} - \begin{pmatrix} C^* \\ -B_1 D_{\cdot 1}^* \end{pmatrix} \tilde{R}^{-1} [D_{\cdot 1} B_1^* \quad C]\right\} \tag{10}$$

and define the 'state feedback' and output injection' matrices as

$$F = \begin{matrix} (m_1-p_2)\updownarrow \\ p_2\updownarrow \\ m_2\updownarrow \end{matrix} \begin{pmatrix} F_{11} \\ F_{12} \\ F_2 \end{pmatrix} = -R^{-1}[D_{1\cdot}^*C_1 + B^*X_\infty], \tag{11}$$

$$H = [\ \underset{(p_1-m_2)}{\overset{}{H_{11}}} \quad \underset{m_2}{H_{12}} \quad \underset{p_2}{H_2}\ ] = -[B_1D_{\cdot 1}^* + Y_\infty C^*]\tilde{R}^{-1}. \tag{12}$$

Remark 1. If there exist X_∞ and Y_∞ satisfying (9) and (10), then it is easily verified that

$$\begin{pmatrix} P_{11}^* \\ P_{12}^* \end{pmatrix}[P_{11} \quad P_{12}] - \begin{pmatrix} I_{m_1} & 0 \\ 0 & 0 \end{pmatrix} = G^*RG, \tag{13}$$

$$\begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix}[P_{11}^* \quad P_{21}^*] - \begin{pmatrix} I_{p_1} & 0 \\ 0 & 0 \end{pmatrix} = \tilde{G}\tilde{R}\tilde{G}^*, \tag{14}$$

where

$$G \overset{s}{=} \left[\begin{array}{c|c} A & B \\ \hline -F & I \end{array}\right], \quad G^{-1} \in H_\infty, \tag{15}$$

$$\tilde{G} \overset{s}{=} \left[\begin{array}{c|c} A & -H \\ \hline C & I \end{array}\right], \quad \tilde{G}^{-1} \in H_\infty. \tag{16}$$

Notice that this is an indefinite spectral factorization problem because if a solution to (6) exists, then it is straightforward to show that R has m_2 positive and m_1 negative eigenvalues, and $\tilde{R}$ has p_2 positive and p_1 negative eigenvalues.

The main result is now stated in terms of the above matrices.

Theorem 1. *For the system described by* (1)–(3) *and satisfying the assumptions* A1–A6:

(a) *There exists an internally stabilizing controller $K(s)$ such that $\|\mathcal{F}_\ell(P, K)\|_\infty < \gamma$ if and only if*

(i) $\gamma > \max(\bar{\sigma}[D_{1111}, D_{1112}], \bar{\sigma}[D_{1111}^*, D_{1121}^*])$ *and*

(ii) *there exist $X_\infty \geqslant 0$ and $Y_\infty \geqslant 0$ satisfying* (9) *and* (10) *respectively and such that $\rho(X_\infty Y_\infty) < \gamma^2$. ($\rho(\cdot)$ denotes the largest eigenvalue.)*

(b) *Given that the conditions of part* (a) *are satisfied, then all rational internally stabilizing controllers $K(s)$ satisfying $\|\mathcal{F}_\ell(P, K)\|_\infty < \gamma$ are given by*

$$K = \mathcal{F}_\ell(K_a, \Phi) \quad \textit{for arbitrary } \Phi \in RH_\infty \quad \textit{such that } \|\Phi\|_\infty < \gamma, \tag{17}$$

where

$$K_a \overset{s}{=} \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right] \tag{18}$$

$$\hat{D}_{11} = -D_{1121}D_{1111}^*(\gamma^2 I - D_{1111}D_{1111}^*)^{-1}D_{1112} - D_{1122}, \tag{19}$$

$\hat{D}_{12} \in \mathbb{R}^{m_2\times m_2}$ and $\hat{D}_{21} \in \mathbb{R}^{p_2\times p_2}$ are any matrices (e.g. Cholesky factors) satisfying

$$\hat{D}_{12}\hat{D}_{12}^* = I - D_{1121}(\gamma^2 I - D_{1111}^*D_{1111})^{-1}D_{1121}^*, \tag{20}$$

$$\hat{D}_{21}^*\hat{D}_{21} = I - D_{1112}^*(\gamma^2 I - D_{1111}D_{1111}^*)^{-1}D_{1112}, \tag{21}$$

and

$$\hat{B}_2 = (B_2 + H_{12})\hat{D}_{12}, \tag{22}$$

$$\hat{C}_2 = -\hat{D}_{21}(C_2 + F_{12})Z, \tag{23}$$

$$\hat{B}_1 = -H_2 + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11}, \tag{24}$$

$$\hat{C}_1 = F_2 Z + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2, \tag{25}$$

$$\hat{A} = A + HC + \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1, \tag{26}$$

where

$$Z = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}. \tag{27}$$

(Note that if $D_{11} = 0$ then the formulae are considerably simplified.)

Remark 2. Assumption A4 will now be removed, i.e. $D_{22} \neq 0$. Suppose K is a stabilizing controller for the case with D_{22} set to zero, and satisfying

$$\left\| \mathcal{F}_\ell\left(P - \begin{pmatrix} 0 & 0 \\ 0 & D_{22} \end{pmatrix}, K\right)\right\|_\infty < \gamma. \tag{28}$$

Then

$$\mathcal{F}_\ell(P, K(I + D_{22}K)^{-1}) = P_{11} + P_{12}K(I + D_{22}K - P_{22}K)^{-1}P_{21} = \mathcal{F}_\ell\left(P - \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix}, K\right). \tag{29}$$

Hence all controllers in this case are given by

$$\tilde{K} = K(I + D_{22}K)^{-1},$$

which leads to

$$\tilde{K} = \mathscr{F}_\ell(\tilde{K}_a, \Phi) \quad \text{for } \Phi \in \mathrm{RH}_\infty, \ \|\Phi\|_\infty < \gamma, \tag{30}$$

where, assuming $\det(I + \hat{D}_{11}D_{22}) \neq 0$,

$$\tilde{K}_a \stackrel{s}{=} \left[\begin{array}{c|c} \hat{A} - \hat{B}(I - M)\hat{D}^{-1}\hat{C} & \hat{B}M \\ \hline \tilde{M}\hat{C} & \hat{D}M \end{array}\right] \tag{31}$$

with

$$M = \left[I + \begin{pmatrix} D_{22} & 0 \\ 0 & 0 \end{pmatrix}\hat{D}\right]^{-1}, \tag{32}$$

$$\tilde{M} = \left[I + \hat{D}\begin{pmatrix} D_{22} & 0 \\ 0 & 0 \end{pmatrix}\right]^{-1}. \tag{33}$$

When $D_{22} \neq 0$ there is a possibility of the feedback system becoming ill-posed due to

$$\det\left(I + D_{22}\tilde{K}(\infty)\right) = 0$$

(or more stringent conditions if we require well-posedness in the face of infinitesimal time delays [19]). Such possibilities need to be excluded from the above parametrization.

Remark 3. Now that the solution to this problem has been stated, its derivation can be constructed using a variety of techniques (several of which are presently under preparation). The formulae were originally derived by the present authors via a new solution to the model-matching problem,

$$\inf_{Q \in H_\infty} \left\| \begin{matrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{matrix} \right\|_\infty$$

and then back substituting to find all controllers. Moreover, relations to several other problems can be found, e.g. deterministic differential games, indefinite factorization, indefinite quadratic forms [18], all-pass dilations, and perhaps, most surprising, risk sensitive stochastic control which is discussed in Section 3.

Remark 4. Since the criterion only involves the H_∞ norm, the discrete-time case can be solved via a bilinear transformation from the unit disk to the half-plane (see for example [10]).

3. Relations to risk sensitive LQG control and maximum entropy

The risk sensitive LQG stochastic control problem for discrete-time systems is formulated as follows (see [16] with some minor changes in notation for compatibility with present setup). Consider the state equation

$$x_t = Ax_{t-1} + B_1 w_t + B_2 u_{t-1}, \tag{34}$$

$$z_t = C_1 x_{t-1} + D_{11} w_t + D_{12} u_{t-1}, \tag{35}$$

$$y_t = C_2 x_{t-1} + D_{21} u_{t-1}, \tag{36}$$

where the process/observation noise w_t is white and Gaussian with unit variance. Let the conventional quadratic cost function be

$$G = x_T^* \Pi x_T + \sum_{t=0}^{T-1} z_t^* z_t. \tag{37}$$

Then the risk sensitive optimal controller minimizes

$$\gamma_T(\theta) = -\frac{2}{\theta T} \log \mathrm{E}[\exp(-\theta G/2)], \tag{38}$$

where E denotes expectation. The scalar parameter θ is called the risk sensitivity parameter: with $\theta \to 0$ giving the standard LQG problem (risk-neutral); $\theta > 0$ being risk-seeking; and $\theta < 0$ being risk-averse. (We will be only concerned with the case $\theta < 0$.)

The case of state-feedback was solved by Jacobson [13] but it was not until Whittle [16] that the general case of imperfect, partially observed states was solved. Whittle showed that the optimal controller is a linear function of a state estimate with the latter generated by a modification of the Kalman filter. Further, Whittle [17] derived a risk-sensitive certainty equivalence principle and a form of separation result. The continuous-time problem has been solved by Bensoussan and Van Schuppen [2] but we will restrict our present discussion to the discrete-time case (results for continuous time can be derived along the lines of Grenander and Szegö [11, §11.8]).

It is not immediately apparent that the H_∞ control problem considered in the present paper is related to the risk-averse LQG problem. In fact, the problems are very closely related, as will now be demonstrated. Two lemmas are firstly required.

Lemma 3.1. *Let z be a normally distributed, zero-mean random vector with covariance matrix V. Then*

$$\mathrm{E}\exp(-\tfrac{1}{2}\theta z^* z) = \begin{cases} [\det(I+\theta V)]^{-1/2} & \text{for } \theta\lambda_{\max}(V) > -1, \\ \infty & \text{for } \theta\lambda_{\max}(V) \leq -1. \end{cases}$$

The proof of Lemma 3.1 is a straightforward exercise and Lemma 3.2 can be derived from the work of Grenander and Szegö [11] and Hannan [12, Chapter III].

Lemma 3.2. *Let R_T be block Toeplitz matrices with i, j-th block entry*

$$\{R_T\}_{ij} = r_{i-r} = r^*_{j-i}, \quad i, j = 1, 2, \dots, T,$$

satisfying $R_T > 0$ for all T and $\Sigma_{-\infty}^{\infty}\bar{\sigma}(r_i) < \infty$. Then

$$\lim_{T\to\infty} \frac{1}{T} \log\det R_T = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\{\det[2\pi f(\lambda)]\}\, \mathrm{d}\lambda$$

where

$$2\pi f(\lambda) = \sum_{k=-\infty}^{\infty} r_k\, \mathrm{e}^{-\mathrm{j}k\lambda}.$$

Let us now evaluate the cost function, $\gamma_T(\theta)$, for a stabilizing LTI controller, with transfer function K. The output z_t will now be a stationary Gaussian process with spectrum

$$f(\lambda) = (1/2\pi)H(\mathrm{e}^{\mathrm{j}\lambda})H(\mathrm{e}^{\mathrm{j}\lambda})^*$$

where $H = \mathscr{F}_\ell(P, K)$. Applying Lemma 3.1 for

$$z^* = [z_0^*, z_1^*, \dots, z_{T-1}^*],$$

with variance given by the Toeplitz matrix, V_T, where $(V_T)_{ij} = v_{i-j}$ and

$$H(\mathrm{e}^{j\lambda})H(\mathrm{e}^{j\lambda})^* = \sum v_k\, \mathrm{e}^{-jk\lambda},$$

gives

$$\mathrm{E}\exp(-\tfrac{1}{2}\theta z^* z) = \det(I + \theta V_T);$$

and then Lemma 3.2 for $R_T = (I + \theta V_T)$ and hence

$$2\pi f(\lambda) = I + \theta H(\mathrm{e}^{j\lambda})H(\mathrm{e}^{j\lambda})^*$$

gives

$$\lim_{T\to\infty}\gamma_T(\theta) = \begin{cases} \dfrac{1}{2\pi\theta}\displaystyle\int_{-\pi}^{\pi}\log\{\det[I + \theta HH^*]\}\,\mathrm{d}\lambda \\ \quad \text{if } \theta\|H\|_\infty^2 > -1, \\ \infty \quad \text{if } \theta\|H\|_\infty^2 < -1. \end{cases}$$

(Note that the cost $x_T^*\Pi x_T$ can be ignored since K is stabilizing and $T\to\infty$.) Hence any LTI optimal controller must be such that

$$\|\mathscr{F}_\ell(P, K)\|_\infty \leq (-\theta)^{-1/2} \quad (\text{for } \theta < 0).$$

Further, the integral expression for $\gamma_T(\theta)$ as $T\to\infty$ is precisely the entropy integral and can be minimized over all LTI controllers meeting the H_∞-norm bound ([1,6,7] and for this particular problem [15]). Indeed, taking $\Phi = 0$ in Theorem 1 gives the optimal controller.

The LTI risk-averse optimal controller therefore minimizes the entropy integral over all controllers that satisfy the H_∞-norm bound. Note that the above argument has not proven that the optimal risk-averse controller as $T\to\infty$ is indeed LTI but does suggest this to be true.

As θ is made more negative, there comes a point beyond which all controllers give infinite cost. The critical value is given by an H_∞-optimal controller which is then maximally risk-averse and further reduction of θ produces "a type of neurotic breakdown" [16, p. 773].

The above analysis has demonstrated a strong link between the maximum entropy, H_∞ controllers and risk-sensitive controllers. Further, in spite of one formulation being probabilistic and the other involving no probability, it turns out that our present derivations have many similarities with those of Whittle [16].

4. Conclusions

The paper has presented a characterization of all controllers that achieve a prescribed bound on the H_∞ norm of a given closed-loop transfer function. Only two Riccati equations need to be solved, each of the same order as the plant. This is a substantial reduction in computation over existing methods, especially when the dimension of $z(t)$ exceeds that of $u(t)$ and the dimension of $w(t)$ exceeds that of $y(t)$.

The relative simplicity of the given solution enables connections to be made with a wealth of other problems (see Remark 3). In particular in Section 3 we have directly shown the connection to Risk-Sensitive LQG controllers via results on maximizing an entropy integral. This gives an equivalence between the above deterministic robust control problem and this stochastic control problem.

An H_∞-optimal controller will be one that achieves the minimum possible value of γ in (6) and has not been considered here. Such optimal controllers can be characterized in a similar way but require intricate manipulations to handle all cases. The limiting value of γ can be reached when $\rho(Y_\infty X_\infty)=\gamma^2$ or when either of the Riccati equations (9) and (10) fail to have solutions, or any combination of these.

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