FLEXIBLE BACKSTEEPING DESIGN FOR TRACKING AND DISTURBANCE ATTENUATION

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SUMMARY

In this paper we consider the benchmark nonlinear control problem and use backstepping to design several active controllers for tracking and disturbance attenuation; these two problems are much more challenging than stabilization. We show that the significant flexibility of backstepping can be exploited to reduce the required control effort and to eliminate the winding problem. (1998 John Wiley & Sons, Ltd.

Key words: nonlinear control; benchmark problem; backstepping; tracking; disturbance attenuation

1. INTRODUCTION

The benchmark problem proposed in Reference 1 considers the stabilization and disturbance attenuation objectives for a translational oscillator with an attached rotational eccentric proof mass, whose actuator torque is the control input. This model, depicted in Figure 1, can be viewed as an important example for nonlinear control design because it exhibits a complex interaction between the translational motion of the cart and the rotational motion of the actuator, which is rather common in mechanical systems. This complex interaction is due to the sinusoidal coupling term, which renders the system non-feedback linearizable and severely limits our control authority over the translational motion of the cart.

In this paper we first show that the stabilization problem is not really a challenging one, unless accompanied by some performance requirements. The reason is that the benchmark nonlinear system can already be viewed as possessing an (almost globally) asymptotically stable equilibrium. In contrast, the tracking and disturbance attenuation problems are much more interesting. Tracking, in particular, requires truly active controllers, which do more than simply dissipate energy. While the passive absorber of Reference 1 and the passivity-based controllers of Reference 2 achieve stabilization and disturbance attenuation with small control effort, we were unsuccessful in our attempts to incorporate the tracking objective into their designs.

An alternative design which can achieve asymptotic tracking is backstepping. As demonstrated in References 1 and 2, this systematic design methodology is applicable to the system at hand, but leads to much more complex controllers which seem to require significantly higher control effort than passivity-based controllers when used for stabilization and disturbance attenuation. So it

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would appear that backstepping can solve a more general class of problems, namely tracking and stabilization with prescribed performance requirements, but does so at a significant penalty in simplicity and control effort.

The main point of this paper is to show that the above tentative conclusion does not take into account the considerable inherent flexibility of backstepping, which allows the designer to choose not only values of gains, but also the form of intermediate nonlinear terms and even of Lyapunov functions, in order to customize the closed-loop system response to meet the required specifications. In particular, we exploit this flexibility to design three backstepping controllers whose differences are structural rather than cosmetic. Then we show that these controllers can achieve stabilization, tracking and disturbance attenuation with levels of control effort comparable to those of passivity-based designs; this is even true of the backstepping controller designed in Reference 1, provided that its gains are chosen more judiciously.

2. STABILIZATION

The system considered here was studied in Reference 3, where it was shown that its behaviour is similar to that of a dual-spin spacecraft exhibiting the *resonance capture phenomenon*. This phenomenon is exhibited in the simplified system when trying to spin-up the rotational proof mass to a desired angular speed: if the available torque is limited in magnitude, the mass can be spun up only to the angular speed \( \sqrt{k/(M + m)} \) which corresponds to the natural frequency of the translational oscillator. Avoiding this capture is a challenging control problem which was solved in Reference 4 using an ingenious heuristic method based on energy arguments and extensive phase-plane analysis.

On the other hand, the stabilization of this system is relatively straightforward. Neglecting friction and gravity (by assuming horizontal plane motion), the equations of motion as stated in Reference 1 are

\[
(M + m)\ddot{q} + kq = -ml(\dot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + F
\]

\[
(I + ml^2)\ddot{\theta} = -mlq\cos\theta + N
\]

where \( F \) is the disturbance force and \( N \) the control torque. Let us now assume that the motion is carried out in a *vertical* plane, so that gravity is restored, and that friction is *not* neglected. Then
the equations of motion become

\[
(M + m)\ddot{q} + F_t \text{sat}_t(q) + kq = -ml\dot{\theta}\cos \theta - \dot{\theta}^2 \sin \theta + F
\]

(3)

\[
(I + ml^2)\ddot{\theta} + N_t \text{sat}_t(\dot{\theta}) + mgl\sin \theta = -ml\ddot{q}\cos \theta + N
\]

(4)

where \(F_t\) and \(N_t\) are the friction force and friction torque coefficients, while \(\text{sat}_t(\cdot)\) and \(\text{sat}_t(\cdot)\) are saturation functions. This is an often-used form of friction, but, as we will see, this specific form is not important for our purpose. What is important is that the friction be present in the form of a first-third-quadrant function of the angular velocity \(\dot{\theta}\).

A suitable Lyapunov function for this system is its total energy

\[
V(q, \dot{q}, \theta, \dot{\theta}) = \frac{1}{2}kq^2 + \frac{1}{2}Mq^2 + mgl(1 - \cos \theta) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{q}^2 + l^2\dot{\theta}^2 + 2q\dot{q}\dot{\theta}\cos \theta)
\]

(5)

Using (3) and (4), the derivative of (5) is computed as follows:

\[
\dot{V} = kq\ddot{q} + M\ddot{q}\sin \theta + l\dot{\theta}\cos \theta - l\dot{\theta}\ddot{q}\sin \theta + N
\]

(6)

Clearly, if \(F \equiv 0\) and \(N \equiv 0\), i.e. in the absence of disturbances and torque actuation, we have \(\dot{V} \leq 0\), which means that the system defined by (3) and (4) is globally stable. Using LaSalle’s invariance theorem, it is easy to see from (3), (4) and (6) that all trajectories converge to one of two equilibrium points: the asymptotically stable \(x = \dot{x} = \theta = \dot{\theta} = 0\) or the unstable \(x = \dot{x} = \theta = \dot{\theta} = \pi\). Hence, when gravity and friction are restored, this system is open-loop-stable.

This calculation suggests that when the motion is carried out in the horizontal plane, a control torque which includes a gravity-like term (such as \(-N_0\sin \theta\)) and a friction-like term (such as \(-N_1 \text{sat}_t(\dot{\theta})\)) would result in similar stability properties. This is the essence of the passive absorber controller designed in Reference 1. Such a controller, however, would have two disadvantages:

1. The decay rate of the closed-loop system cannot be made faster than a certain upper bound. This fact is observed through simulations in Reference 1 and confirmed analytically through a root-locus argument in Reference 2, where the authors also design another passivity-based stabilizing controller which overcomes this problem.

2. To the best of our knowledge, the structure of this controller cannot be modified to accommodate an asymptotic tracking objective, where the position of the cart is required to track an appropriately defined reference signal.

Both of these problems are easily overcome by using backstepping to design active controllers.

3. A NON-WINDING DESIGN FOR TRACKING

The so-called winding problem for the system of Figure 1 occurs when the controller views the angular displacement of the proof mass as taking values on \(\mathbb{R}\) and insists on regulating it to zero.
This means that even if the system is initially at rest but with the actuator angle ‘wound up’ to $2\pi$ rather than 0, such a controller would ‘unwind’ the actuator, thereby unnecessarily exciting the system and leading to a waste of control effort.

In this section, we present in detail a straightforward backstepping scheme which eliminates the winding problem and yields tracking and disturbance attenuation results representative of all the schemes presented in this paper.

To proceed with this design, let us first switch to the normalized dimensionless co-ordinates of Reference 1, in which the system defined by (1) and (2) is expressed as

$$\ddot{\xi} + \xi = \varepsilon(\dot{\theta}^2 \sin \theta - \dot{\theta} \cos \theta) + w$$

$$\ddot{\theta} = -\varepsilon \dot{\xi} \cos \theta + u$$

or, equivalently,

$$(1 - \varepsilon^2 \cos^2 \theta)\ddot{\xi} + \xi = \varepsilon(\dot{\theta}^2 \sin \theta - \cos \theta u) + w$$

$$(1 - \varepsilon^2 \cos^2 \theta)\ddot{\theta} = \varepsilon \cos \theta (\xi - \varepsilon \dot{\theta}^2 \sin \theta \cos \theta) + u$$

where $\xi$ is the cart displacement, $\theta$ is the proof mass angle, $w$ is the external disturbance force and $u$ is the actuator torque. Furthermore, we assume $|w| \leq M_d$ with $M_d$ being a possibly unknown positive constant.

Suppose now that the primary control objective is to track a small reference signal $\xi_{\text{ref}}$ (whose derivatives $\dot{\xi}_{\text{ref}}$ and $\ddot{\xi}_{\text{ref}}$ are known and small) with the output $\xi$, while attenuating the disturbance $w$. Of course, the physical structure of the system does not allow tracking of arbitrary reference signals. It is clear, for example, that a constant non-zero $\xi_{\text{ref}}$ cannot be tracked. In this paper we consider only oscillatory signals of small amplitude.

A natural approach to the tracking problem is to use a backstepping design 5. This approach was adopted for the stabilization problem in Reference 6, where it was shown that the change of co-ordinates

$$x_1 = \xi + \varepsilon \sin \theta, \quad x_2 = \dot{\xi} + \varepsilon \dot{\theta} \cos \theta, \quad x_3 = \theta, \quad x_4 = \dot{\theta}$$

and the feedback

$$u = (1 - \varepsilon^2 \cos^2 \theta)v - (\varepsilon \dot{\xi} \cos \theta - \varepsilon^2 \dot{\theta}^2 \cos \theta \sin \theta)$$

transform (7) into the cascade form:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \varepsilon \sin x_3 + w$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = v$$

The only thing missing for satisfying all the prerequisites for the backstepping design of a tracking controller is the definition of suitable reference signals for the states $x_1$–$x_4$. The reference output $\dot{\xi}_{\text{ref}} = A_r \sin \omega_r t$ and its derivatives $\dot{\xi}_{\text{ref}} = A_r \omega_r \cos \omega_r t$ and $\ddot{\xi}_{\text{ref}} = -A_r \omega_r^2 \sin \omega_r t$ are assumed to be
known. Then, to find the reference signal for $\theta$, we introduce an intermediate variable $p(t) = -((1 - \omega_r^2)/\varepsilon \omega_r^2) \xi_{\text{ref}}$ and compute (off-line)

$$\theta_{\text{ref}} = \arcsin (p) = \arcsin \left(-A_r \frac{(1 - \omega_r^2)}{\varepsilon \omega_r^2} \sin \omega_r t \right)$$  \hspace{1cm} (12)$$

$$\dot{\theta}_{\text{ref}} = \frac{\dot{p}}{\cos \theta_{\text{ref}}} = \frac{(1 - \omega_r^2)}{\varepsilon \omega_r^2 \cos \theta_{\text{ref}}} \xi_{\text{ref}} = \frac{A_r (1 - \omega_r^2)}{\varepsilon \omega_r^2 \cos \theta_{\text{ref}}} \cos \omega_r t$$  \hspace{1cm} (13)$$

$$\ddot{p} = \frac{\ddot{\theta}_{\text{ref}} \cos \theta_{\text{ref}} - \dot{\theta}_{\text{ref}}^2 \sin \theta_{\text{ref}}}{\cos \theta_{\text{ref}}} = \ddot{\theta}_{\text{ref}} \cos \theta_{\text{ref}} - \frac{p}{(\varepsilon \omega_r^2)^2 \cos^2 \theta_{\text{ref}}} \xi_{\text{ref}}^2$$  \hspace{1cm} (14)$$

from which $\ddot{\theta}_{\text{ref}}$ will be

$$\ddot{\theta}_{\text{ref}} = \frac{\ddot{p}}{\cos \theta_{\text{ref}}} - \frac{p \dot{p}^2}{\cos^3 \theta_{\text{ref}}}$$

$$= -\frac{1}{\varepsilon \cos \theta_{\text{ref}}} \left(\dot{\xi}_{\text{ref}} + \xi_{\text{ref}} \right) + \frac{p}{(\varepsilon \omega_r^2)^2 \cos \theta_{\text{ref}}} \dot{\xi}_{\text{ref}}$$

$$= -\frac{A_r (1 - \omega_r^2)}{\varepsilon \omega_r^2 \cos \theta_{\text{ref}}} \sin \omega_r t - \frac{A_r^3 (1 - \omega_r^3)}{\varepsilon^4 \omega_r^4 \cos^4 \theta_{\text{ref}}} \sin \omega_r t \cos^2 \omega_r t$$  \hspace{1cm} (15)$$

Clearly, the reference signal $\xi_{\text{ref}}$ satisfies

$$\dot{\xi}_{\text{ref}} + \xi_{\text{ref}} = -\varepsilon \frac{d^2}{dt^2} \sin \theta_{\text{ref}}$$  \hspace{1cm} (16)$$

Using the transformation of (9) together with (12)–(16), we can compute the reference signals $x_{1\text{ref}}$–$x_{4\text{ref}}$ as follows:

$$x_{1\text{ref}} = \xi_{\text{ref}} + \varepsilon p$$

$$x_{2\text{ref}} = \dot{\xi}_{\text{ref}} + \varepsilon \dot{p}$$

$$x_{3\text{ref}} = \theta_{\text{ref}}$$

$$x_{4\text{ref}} = \dot{\theta}_{\text{ref}}$$  \hspace{1cm} (17)$$

The backstepping design then proceeds as follows:

**Step 1:** We start with the error variable

$$z_1 = x_1 - x_{1\text{ref}}$$  \hspace{1cm} (18)$$

whose derivative can be expressed by using the first two equations in (11) and (17) as

$$\dot{z}_1 = x_2 - x_{2\text{ref}}$$  \hspace{1cm} (19)$$
In (19), we view $x_2$ as the virtual control, and introduce the error variable $z_2 = x_2 - \alpha_1$, where $\alpha_1$ is the first stabilizing function, yet to be determined. Then we can represent $\dot{z}_1$ as

$$\dot{z}_1 = \alpha_1 + z_2 - x_{2\text{ref}}$$  \hspace{1cm}  (20)

In order to design $\alpha_1$, we choose the partial Lyapunov function $V_1 = \frac{1}{2}z_1^2$ and compute its time derivative of $V_1$ along the solutions of (20):

$$\dot{V}_1 = z_1 \dot{z}_1$$

$$\hspace{2cm} = z_1 (\alpha_1 + z_2 - x_{2\text{ref}})$$  \hspace{1cm}  (21)

The choice of

$$\alpha_1 = -c_1 z_1 + x_{2\text{ref}}$$  \hspace{1cm}  (22)

yields $\dot{V}_1 = -c_1 z_1^2 + z_1 z_2$, which is negative-definite in $z_1$ when $z_2 = 0$. In addition, we obtain

$$\dot{z}_1 = -c_1 z_1 + z_2$$  \hspace{1cm}  (23)

**Step 2:** According to the computation in Step 1, driving $z_2$ to zero will ensure that $\dot{V}_1$ is negative-definite in $z_1$. Therefore, we need to modify the Lyapunov function to include the error variable $z_2$. For our derivation, we choose the modified Lyapunov function as

$$V_2 = \frac{1}{2}(z_1^2 + z_2^2)$$  \hspace{1cm}  (24)

Then using (11), (16) and (17), we rewrite $\dot{z}_2$ as

$$\dot{z}_2 = \dot{x}_2 - \dot{z}_1$$

$$= -x_1 + \varepsilon \sin x_3 + c_1 \dot{z}_1 - \dot{x}_{2\text{ref}} + w$$

$$= -x_1 + \varepsilon \sin x_3 + c_1 (-c_1 z_1 + z_2) - \dot{x}_{2\text{ref}} - \dot{z}_1 + \xi_{\text{ref}} + w$$

$$= -x_1 + \varepsilon \sin x_3 + c_1 z_2 - c_1^2 z_1 + \xi_{\text{ref}} + w$$

$$= -z_1 + \varepsilon (\sin x_3 - \sin x_{3\text{ref}}) + c_1 z_2 - c_1^2 z_1 + w$$  \hspace{1cm}  (25)

In this equation, we view $\sin x_3$ as the virtual control. This is a departure from the usual backstepping designs which only employ state variables as virtual controls. In this case, however, this simple modification is not only dictated by the structure of the system, but it also yields significant improvements in closed-loop system response. The new error variable is $z_3 = \sin x_3 - \alpha_2$, and $\alpha_2$ is yet to be computed. Then (25) becomes

$$\dot{z}_2 = -z_1 + \varepsilon (\alpha_2 + z_3 - \sin x_{3\text{ref}}) + c_1 z_2 - c_1^2 z_1 + w$$  \hspace{1cm}  (26)

and the time derivative of $V_2$ along the solutions of (23) and (26) is

$$\dot{V}_2 = z_1 \dot{z}_1 + z_2 \dot{z}_2$$

$$\hspace{2cm} = -c_1 z_1^2 + z_1 z_2 + z_2 \left[ -z_1 + \varepsilon (\alpha_2 + z_3 - \sin x_{3\text{ref}}) + c_1 z_2 - c_1^2 z_1 \right] + z_2 w$$

$$\hspace{2cm} = -c_1 z_1^2 + z_2 \left[ \varepsilon (\alpha_2 - x_{3\text{ref}}) + c_1 z_2 - c_1^2 z_1 \right] + z_2 w$$

$$\hspace{2cm} = -c_1 z_1^2 - c_2 z_2^2 + \varepsilon z_2 z_3 + z_2 \left[ c_2 z_2 + \varepsilon (\alpha_2 - x_{3\text{ref}}) + c_1 z_2 - c_1^2 z_1 \right] + z_2 w$$  \hspace{1cm}  (27)
From (27), the choice of
\[ x_2 = \sin x_{3\text{ref}} + \frac{1}{\varepsilon} [c_1^2 z_1 - (c_1 + c_2) z_2] \] (28)
renders
\[ \dot{z}_2 = -c_2 z_2 - z_1 + \varepsilon z_3 + w \] (29)
and
\[ \dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + \varepsilon z_2 z_3 + z_2 w \] (30)
Hence, we obtain \( \dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 w \) when \( z_3 = 0 \).

One of the benefits of using \( \sin x_3 \) instead of \( x_3 \) as the virtual control is that the resulting controller will be free from the winding problem, since \( \sin x_3 \) is a periodic function of \( x_3 \). The controller will attempt to asymptotically regulate \( \sin x_3 \) to zero, but will not be able to distinguish between \( x_3 = 0 \) and \( 2\pi n \) where \( n \) is an integer. In this way, the controller will drive \( x_3 \) to either \( x_3 = 2\pi n \) or \( (2\pi n + 1)\pi \) which are both open-loop-stable (without gravity).

**Step 3:** Similar to the previous steps, we will design the stabilizing function \( \alpha_2 \) in this step. To achieve that, we first compute \( \dot{z}_3 \) as follows:
\[ \dot{z}_3 = x_4 \cos x_3 - \dot{x}_2 \triangleq x_4 \cos x_3 - \alpha_2 + d_2 w \] (31)
with
\[ d_2 \triangleq \frac{(c_1 + c_2)}{\varepsilon} \]
and
\[ \alpha_2 = \dot{x}_2 + d_2 w \]
\[ = \dot{x}_2 - \frac{\partial x_2}{\partial z_2} w \]
\[ = \left[ \begin{array}{c} \frac{\partial x_2}{\partial x_{3\text{ref}}} \frac{\partial x_2}{\partial z_1} \frac{\partial x_2}{\partial z_2} \\ \frac{\partial x_2}{\partial x_{3\text{ref}}} \frac{\partial x_2}{\partial z_1} \frac{\partial x_2}{\partial z_2} \end{array} \right] \left[ \begin{array}{c} x_{4\text{ref}} \\ -c_1 z_1 + z_2 \\ -c_2 z_2 - z_1 + \varepsilon z_3 + w \end{array} \right] - \frac{\partial x_2}{\partial z_2} w \]
\[ = \left[ \begin{array}{c} \frac{\partial x_2}{\partial x_{3\text{ref}}} \frac{\partial x_2}{\partial z_1} \frac{\partial x_2}{\partial z_2} \\ \frac{\partial x_2}{\partial x_{3\text{ref}}} \frac{\partial x_2}{\partial z_1} \frac{\partial x_2}{\partial z_2} \end{array} \right] \left[ \begin{array}{c} x_{4\text{ref}} \\ -c_1 z_1 + z_2 \\ -c_2 z_2 - z_1 + \varepsilon z_3 \end{array} \right] \] (32)

For any signal \( z \), we use the notation \( \alpha \) to denote the part of its time derivative \( \dot{z} \) which can be computed analytically from known quantities. This is done to emphasize the fact that our controller does not use numerical differentiation. In (31), for example, \( \dot{z}_2 \) contains the unknown disturbance term \( d_2 w \); this term is excluded from \( \alpha_2 \), which in turn is computed analytically in (32).
In (31), we use \( x_4 \cos x_3 \) as the virtual control, in what might seem as yet another unconventional choice. Then we define the error variable \( z_4 = x_4 \cos x_3 - \alpha_3 \), and rewrite \( \dot{z}_3 \) as

\[
\dot{z}_3 = \alpha_3 + z_4 - \bar{\alpha}_2 + d_2 w
\]  

(33)

Therefore, along the solutions of (23), (29) and (33), we can express the time derivative of the partial Lyapunov function \( V_3 = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2) \) as

\[
\dot{V}_3 = z_1 \dot{z}_1 + z_2 \dot{z}_2 + z_3 \dot{z}_3
\]

\[
= -c_1 z_1^2 - c_2 z_2^2 + c_3 z_3^2 + z_3 (x_4 \cos x_3 - \alpha_2 + d_2 w) + z_2 w
\]

\[
= -c_1 z_1^2 - c_2 z_2^2 + c_3 z_3^2 + z_3 (\alpha_3 + z_4 - \bar{\alpha}_2) + z_2 w + d_2 z_3 w
\]

\[
= -c_1 z_1^2 - c_2 z_2^2 + z_3 (\alpha_3 + \varepsilon z_4 - \bar{\alpha}_2) + z_3 z_4 + z_2 w + d_2 z_3 w
\]  

(34)

From (34), the selection

\[
\alpha_3 = \bar{\alpha}_2 - c_3 z_3 - \varepsilon z_2
\]  

(35)

yields

\[
\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + z_2 w + d_2 z_3 w
\]  

(36)

and

\[
\dot{z}_3 = -c_3 z_3 + z_4 - \varepsilon z_2 + d_2 w
\]  

(37)

**Step 4:** We first modify the Lyapunov function \( V_3 \) from Step 3 to include the error variable \( z_4 \)

\[
V = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2 + z_4^2)
\]  

(38)

Then repeating the process from previous steps, we compute

\[
\dot{z}_4 = v \cos x_3 - x_4^2 \sin x_3 - \dot{\bar{\alpha}}_3 = v \cos x_3 - x_4^2 \sin x_3 - \bar{\alpha}_3 - d_3 w
\]  

(39)

where we have

\[
d_3 = -\varepsilon + \frac{c_1^2 + c_2^2 + c_3^2}{\varepsilon} + d_2 (c_1 + c_2 + c_3)
\]

and

\[
\bar{\alpha}_3 = \dot{\bar{\alpha}}_3 - d_3 w
\]

\[
= \dot{\bar{\alpha}}_3 - \left[ \frac{\partial \bar{\alpha}_3}{\partial z_2} + d_2 \frac{\partial \bar{\alpha}_3}{\partial z_3} \right] w
\]
\[
\begin{bmatrix}
\frac{\partial x_3}{\partial x_{3, \text{ref}}} & \frac{\partial x_3}{\partial x_{4, \text{ref}}} & \frac{\partial x_3}{\partial z_1} & \frac{\partial x_3}{\partial z_2} & \frac{\partial x_3}{\partial z_3}
\end{bmatrix}
\begin{bmatrix}
-\left(\ddot{\xi} + \dot{\xi}_\text{ref}\right) - \left(x_{4, \text{ref}} \cos \theta_\text{ref}\right)^2 \sin \theta_\text{ref} \\
\varepsilon \cos \theta_\text{ref} \\
\cos^3 \theta_\text{ref}
\end{bmatrix}
\begin{bmatrix}
- c_1 z_1 + z_2 \\
- c_2 z_2 - z_1 + \varepsilon z_3 + w \\
- c_3 z_3 + z_4 - \varepsilon z_2 + d_2 w
\end{bmatrix}
\]

\[
- \frac{\partial x_3}{\partial z_2} w - d_2 \frac{\partial x_3}{\partial z_3} w
\]

Therefore, along the solutions of (23), (29), (37) and (39), the time derivative of \( V \) is

\[
\dot{V} = (z_1 \dot{z}_1 + z_2 \dot{z}_2 + z_3 \dot{z}_3 + z_4 \dot{z}_4)
\]

\[
= z_1 (-c_1 z_1 + z_2) + z_2 (-c_2 z_2 - z_1 + \varepsilon z_3 + w) + z_3 (-c_3 z_3 + z_4 - \varepsilon z_2 + d_2 w)
\]

\[
+ z_4 (v \cos x_3 - x_4^2 \sin x_3 - \bar{x}_3 - d_3 w)
\]

\[
= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 z_4 + z_4 (v \cos x_3 - x_4^2 \sin x_3 - \bar{x}_3) + z_2 w + d_2 z_3 w - d_3 z_4 w
\]

\[
= -c_1 z_1^2 - \frac{c_2}{2} z_2^2 - \frac{c_3}{2} z_3^2 + z_4 (v \cos x_3 - x_4^2 \sin x_3 - \bar{x}_3 + z_3) - \frac{c_2}{2} z_2^2 - \frac{c_3}{2} z_3^2 + z_2 w
\]

\[
+ d_2 z_3 w - d_3 z_4 w
\]

\[
\leq -c_1 z_1^2 - \frac{c_2}{2} z_2^2 - \frac{c_3}{2} z_3^2 + \frac{c_2}{2} z_2^2 + |z_2| M_d - \frac{c_3}{2} z_3^2 + |z_3| |d_2| M_d
\]

\[
+ z_4 (v \cos x_3 - x_4^2 \sin x_3 - \bar{x}_3) + |d_3| |z_4| M_d
\]

\[
\leq -c_1 z_1^2 - \frac{c_2}{2} z_2^2 - \frac{c_3}{2} z_3^2 + z_4 (v \cos x_3 - x_4^2 \sin x_3 - \bar{x}_3 + z_3) - \frac{c_2}{2} \left(\frac{|z_2|}{c_2} \frac{1}{M_d}\right)^2
\]

\[
+ \frac{1}{2c_2} M_d^2 + \frac{d_2^2}{2c_3} M_d^2 - \frac{c_3}{2} \left(\frac{|z_3|}{c_3} \frac{d_2}{M_d}\right)^2 + \frac{d_3^2}{2c_4} M_d^2 - \frac{c_4}{2} \left(\frac{|z_4|}{c_4} \frac{d_2}{M_d}\right)^2 + \frac{c_4}{2} z_4^2
\]

\[
\leq -c_1 z_1^2 - \frac{c_2}{2} z_2^2 - \frac{c_3}{2} z_3^2 + z_4 \left(v \cos x_3 - x_4^2 \sin x_3 - \bar{x}_3 + z_3 + \frac{c_4}{2} z_4\right) + d_4 M_d^2
\]
with

\[ d_4 \triangleq \frac{1}{2c_2} + \frac{d_2^2}{2c_3} + \frac{d_3^3}{2c_4} \]

In the above expression, our choice of \( v \) is \( v = (1/\cos x_3) ( -c_4 z_4 - z_3 + x_4^2 \sin x_3 + \bar{z}_3 ) \). However, since \( x_3 = \pi/2 \) would yield a division by zero, we modify our control law to

\[
v = \begin{cases} \frac{1}{\cos x_3} ( -c_4 z_4 - z_3 + x_4^2 \sin x_3 + \bar{z}_3 ) & \text{for } |\cos x_3| \geq a \\ \frac{1}{a} \cos x_3 ( -c_4 z_4 - z_3 + x_4^2 \sin x_3 + \bar{z}_3 ) & \text{for } |\cos x_3| < a \end{cases}
\]

where \( a \) is a small positive constant. With this choice, we have

\[
V \leq \begin{cases} -c_1 z_1^2 - \frac{c_2}{2} z_2^2 - c_3 z_3^2 - \frac{c_4}{2} z_4^2 + d_4 M_d^2 & \text{for } |\cos x_3| \geq a \\ -c_1 z_1^2 - \frac{c_2}{2} z_2^2 - c_3 z_3^2 - \frac{c_4}{2a^2} z_4^2 \cos^2 x_3 + d_4 M_d^2 \\ + z_4 \left( 1 - \frac{1}{2a^2} \cos^2 x_3 \right) ( -x_4^2 \sin x_3 - \bar{z}_3 + z_3 ) & \text{for } |\cos x_3| < a \end{cases}
\]

Clearly, when \( a \ll 1, x_3 \) must be very close to \( \pm \pi/2 \) in order to satisfy \(|\cos x_3| < a\). From a control designer’s point of view, however, the set \( \mathcal{S} \triangleq \{ x \in \mathbb{R}^4 : |\cos x_3| < a \} \) with \( a \ll 1 \) is the region where the authority of the controller over the translational part of the system is virtually non-existent. Therefore, the designer’s task is to ensure that the system trajectories will not enter this region in the first place, and as we will see below, our controller is successful in satisfying this requirement. In this case, we can conclude \( V \leq 0 \) if \( c_m \|(z_1(t), z_2(t), z_3(t), z_4(t))\|^2 \geq d_4 M_d^2 \) with \( c_m \triangleq \min \{ c_1, c_2/2, c_3/2, c_4/2 \} \), which, in view of (38), is equivalent to \( 2c_m V \geq d_4 M_d^2 \). Hence, we have \( V_{\max} \leq \{ (d_4/2c_m) M_d^2, V(0) \} \), which implies that

\[
\lim_{t \to \infty} \sup V(t) \leq \frac{d_4}{2c_m} M_d^2 \Rightarrow \lim_{t \to \infty} \|(z_1(t), z_2(t), z_3(t), z_4(t))\| \leq \sqrt{\frac{d_4}{2c_m}} M_d
\]

Using the definition of \( d_2, d_3, \) and \( d_4 \), we can rewrite \( \sqrt{d_4/2c_m} \) as

\[
\sqrt{\frac{d_4}{2c_m}} = \sqrt{\frac{1}{2c_1 c_m} \left( c_1 + c_4 \right)^2 + \frac{\left[ -\varepsilon^2 + (c_1^2 + c_2^2 + c_1 c_2) - (c_1 + c_2)(c_1 + c_2 + c_3) \right]^2}{2 \varepsilon^2 c_4 c_m}}
\]

From this equation, we see that in order to achieve disturbance attenuation, it is sufficient to choose

1. a large \( c_2 \) to reduce the value of \( 1/2c_2c_m \),
2. a larger \( c_3 \) to reduce the value of \( (c_1 + c_2)^2/2c_3c_m \),
3. an even larger \( c_4 \) to reduce the value of
\[
\frac{\left[ -\varepsilon^2 + (c_1^2 + c_2^2 + c_1 c_2) - (c_1 + c_2)(c_1 + c_2 + c_3) \right]^2}{2 \varepsilon^2 c_4 c_m}
\]
In this way, $\sqrt{d_4/2e_m}$ can be made arbitrarily small, which, in view of (44), implies that the effect of the disturbance on the closed loop can be arbitrarily attenuated. The following simulation results also confirm that the disturbance attenuation and tracking objectives are achieved:

1. Figure 3 corresponds to the case where the system is completely driven by the disturbance force $F(t) = 8 \sin 2t$ with no control effort being applied (i.e. $N = 0$).
2. Figure 4 depicts stabilization and disturbance attenuation results by the passive absorber. Here, we adopted

$$N = - N_k \tanh(r\theta) - K_s \sin \theta$$

(46)

given in Reference 1 as the passive absorber.

- In Figure 4, the gains of the passive absorber were chosen as $r = 0.0107$, $N_k = 9$ and $K_s = 0.1$.

The initial conditions for the above simulations were $x_1 = x_2 = x_3 = x_4 = 0$, and the disturbances were sinusoidal forces.

3. Figure 2 shows tracking of a sinusoidal reference signal $0.1 \sin 1.5t$ with the above backstepping controller under the same choice of $c_1 = c_2 = c_4 = 0.0001$, $c_3 = 0.7$ and $a = 0$. The initial conditions for tracking were also $x_1 = x_2 = x_3 = x_4 = 0$.

4. Table I presents the other cases where stabilization and disturbance attenuation are achieved by the above passive absorber and backstepping controller with the gains used in Figures 2–4.

The backstepping controller can achieve almost identical performance as the passive absorber (shown in Figure 4), for both stabilization and disturbance attenuation, when the design constants are chosen as $c_1 = c_2 = c_4 = 0.0001$, $c_3 = 0.7$ and $a = 0$. Here, our choice of the design constants $c_2$, $c_3$ and $c_4$ is not consistent with the guidelines 1–3 given below (45) for the disturbance attenuation, which are, in general, too conservative. The design constants $c_1$–$c_4$ were first chosen according to guidelines, but then we used the iterative optimization routines of MATLAB’s Nonlinear Control Design Toolbox to obtain a new set of values for $c_1$–$c_4$. With this new set of values, less control effort was used without sacrificing tracking performance or disturbance attenuation compared to the initial choices of $c_1$–$c_4$. Furthermore, this performance comparison between these two controllers is also valid when we change the frequencies of disturbance signals. There are no comparisons to the passive absorber controller for the tracking case, because we were unable to design a tracking controller based on the same energy dissipation idea.

4. DESIGN FLEXIBILITY

Now that we have designed one backstepping controller which satisfies the control objectives of stabilization, tracking and disturbance attenuation, let us explore other avenues of design by taking advantage of the many flexibilities of the backstepping methodology. The key point here is that these flexibilities allow the designer to use in his/her design any initial stabilizing function or any Lyapunov function suitable for the system at hand. The performance of these two controllers designed in this section is quite similar to that of the previous one. Thus, no additional simulation results are presented.
Table I. Tracking and disturbance attenuation by the backstepping controller (B) of (42) and disturbance attenuation by the passive absorber (P) of (46). The data are recorded from 0 to 400 s

<table>
<thead>
<tr>
<th>Controller</th>
<th>B</th>
<th>B</th>
<th>B</th>
<th>B</th>
<th>B</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reference signal</td>
<td>$0 \cdot \sin t$</td>
<td>$0 \cdot \sin t$</td>
<td>$0 \cdot \sin 1 \cdot 5t$</td>
<td>$0 \cdot \sin 0 \cdot 8t$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Disturbance signal</td>
<td>0</td>
<td>$0 \cdot \sin 2t$</td>
<td>$0 \cdot \sin 2t$</td>
<td>0</td>
<td>$\sin 2t$</td>
<td>$\sin 2t$</td>
</tr>
<tr>
<td>Initial range of actuator motion (deg)</td>
<td>$\pm 0 \cdot 8$</td>
<td>$\pm 0 \cdot 55$</td>
<td>$\pm 43 \cdot 5$</td>
<td>$\pm 33 \cdot 5$</td>
<td>$\pm 7 \cdot 5$</td>
<td>$-46 \sim 7 \cdot 5$</td>
</tr>
<tr>
<td>Final range of actuator motion (deg)</td>
<td>$\pm 10 \cdot 2$</td>
<td>$\pm 0 \cdot 24$</td>
<td>$\pm 34$</td>
<td>$\pm 33 \cdot 5$</td>
<td>$\pm 24$</td>
<td>$-21 \sim 2 \cdot 3$</td>
</tr>
<tr>
<td>Initial range of tracking error (m)</td>
<td>$\pm 10 \cdot 2$</td>
<td>$\pm 6 \cdot 10 \cdot 3$</td>
<td>$\pm 1 \cdot 3 \cdot 10 \cdot 2$</td>
<td>$\pm 1 \cdot 4 \cdot 10 \cdot 2$</td>
<td>$\pm 1 \cdot 5 \cdot 10 \cdot 2$</td>
<td>$\pm 1 \cdot 5 \cdot 10 \cdot 2$</td>
</tr>
<tr>
<td>Final range of tracking error (m)</td>
<td>$\pm 10 \cdot 4$</td>
<td>$\pm 3 \cdot 10 \cdot 3$</td>
<td>$\pm 4 \cdot 10 \cdot 2$</td>
<td>$\pm 2 \cdot 10 \cdot 3$</td>
<td>$\pm 9 \cdot 10 \cdot 3$</td>
<td>$\pm 9 \cdot 10 \cdot 2$</td>
</tr>
<tr>
<td>Initial range of the cart motion (m)</td>
<td>$\pm 1 \cdot 4 \cdot 10 \cdot 3$</td>
<td>$\pm 1 \cdot 4 \cdot 10 \cdot 3$</td>
<td>$\pm 0 \cdot 19$</td>
<td>$\pm 4 \cdot 6 \cdot 10 \cdot 2$</td>
<td>$\pm 1 \cdot 5 \cdot 10 \cdot 2$</td>
<td>$\pm 1 \cdot 5 \cdot 10 \cdot 2$</td>
</tr>
<tr>
<td>Final range of the cart motion (m)</td>
<td>$\pm 1 \cdot 0 \cdot 5$</td>
<td>$\pm 7 \cdot 1 \cdot 0 \cdot 3$</td>
<td>$\pm 0 \cdot 1$</td>
<td>$\pm 4 \cdot 6 \cdot 10 \cdot 2$</td>
<td>$\pm 7 \cdot 1 \cdot 0 \cdot 3$</td>
<td>$\pm 8 \cdot 1 \cdot 0 \cdot 3$</td>
</tr>
</tbody>
</table>
Before we start, recall that the cascade form of the system is:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon \sin x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= v
\end{align*}
\]  

(47)

4.1. Method 1

Here we will follow the same path taken in Reference 6, where the backstepping process was initiated by viewing \( x_3 \) in the second equation of (47) as the virtual control and choosing \(-c_0 \arctan x_2\) as the corresponding stabilizing function. This choice renders the origin of the subsystem consisting of the first two equations in (47) globally asymptotically stable when \( x_3 = -c_0 \arctan x_2 \).

Let us then denote by \( x_0 \) the ‘initial’ stabilizing function for \( x_3 \). This function \( x_0 \) can be anything which globally asymptotically stabilizes the origin of the above-mentioned two-dimensional subsystem, such as \( x_0 = -c_0 \arctan x_2/(1 + x_3^2) \) or \( x_0 = -c_0 x_2/(1 + x_3^2) \). Of course, the choice of \( x_0 \) will have a significant effect on the overall performance, since different choices will result in structurally different controllers. The question thus becomes which initial stabilizing function will result in the best performance. As an example, we first present the backstepping controller given in References 1 and 6:

\[
v = -c_2(x_4 - x_1) + \ddot{x}_1 - \frac{p_1}{p_2} (x_3 - x_0)
\]  

(48)
Figure 3. The behaviour of our system under the disturbance \( F(t) = 8 \sin 2t \) when there is no control effort (i.e. \( N = 0 \)). The time scale of the disturbance signal plot is different for clarity.

where \( \bar{z}_1 \) is given by

\[
\bar{z}_1 = \begin{bmatrix}
\frac{\partial z_0}{\partial x_1} & \frac{\partial z_1}{\partial x_2} & \frac{\partial z_1}{\partial x_3}
\end{bmatrix}
\begin{bmatrix}
x_2 \\
-x_1 + \varepsilon \sin x_3 \\
x_4
\end{bmatrix}
\] (49)

and

\[
\alpha_1 = \begin{bmatrix}
\frac{\partial z_0}{\partial x_2} (-x_1 + \varepsilon \sin x_3) + \frac{\varepsilon p_0 x_2 (\sin x_0 - \sin x_3)}{p_1 (x_3 - x_0)} - c_1(x_3 - x_0) & \text{for } x_3 - x_0 \neq 0 \\
\frac{\partial z_0}{\partial x_2} (-x_1 + \varepsilon \sin x_3) - \frac{\varepsilon p_0 x_2 \cos x_0}{p_1} & \text{for } x_3 - x_0 = 0
\end{bmatrix}
\] (50)

The controller (48)–(50) renders the derivative of the Lyapunov function

\[
V = \frac{p_0}{2} (z_1^2 + z_2^2) + \frac{p_1}{2} (x_3 + c_0 \arctan z_2)^2 + \frac{p_2}{2} (x_4 - x_1)^2
\] (51)

eightsevenskip

negative-semidefinite in \( z_1, z_2, z_3 \) and \( z_4 \):

\[
\dot{V} = -\varepsilon p_0 x_2 \sin (c_0 \arctan x_2) - p_1 c_1 (x_3 + c_0 \arctan x_2)^2 - p_2 c_2 (x_4 - x_1)^2
\] (52)

If we use \( x_0 = -c_0 \arctan x_2 \) as our initial stabilizing function and the controller (48)–(50), we end up with the scheme designed in References 1 and 6.
4.2. Method 2

As we already mentioned in Method 1, backstepping is flexible with respect to the choice of initial stabilizing function. In the following, we will use another feature of it, namely the fact that backstepping can also accommodate many different choices of Lyapunov function.

Returning to (47) and defining $z_1 = x_1$ and $z_2 = x_2$ as the error variables, we can represent the first two error equations as

\begin{align}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -z_1 + \varepsilon \sin x_3
\end{align}

In (54) we view $x_3$ as the virtual control, and introduce $z_3 = x_3 - x_0$ as the new error variable:

\begin{align}
x_3 &= x_0 + z_3 \\
\dot{z}_2 &= z_1 + \varepsilon \sin (x_0 + z_3)
\end{align}

where $x_0$ is yet to be determined.

To drive $z_1$ and $z_2$ to zero, we choose the partial Lyapunov function as follows:

\[ V_1 = \frac{p_0}{2} (z_1^2 + z_2^2) \]

Using (53) and (56), we have

\[ \dot{V}_1 = p_0 (z_1 \dot{z}_1 + z_2 \dot{z}_2) = \varepsilon p_0 z_2 \sin (x_0 + z_3) \]
The initial stabilizing function $\alpha_0 = -c_0z_2/(1 + z_2^2)$ renders $\dot{V}_1$ negative-semidefinite in $z_1$ and $z_2$ when $z_3 = 0$.

Now, we want to include $z_3$ into our Lyapunov function. Instead of the usual quadratic choice $z_3^2$, we use a trigonometric term, which is positive-definite and monotonically increasing with $|z_3|$ on the set $z_3 \in [-2\pi, 2\pi]$:

$$V_2 = \frac{p_0}{2} (z_1^2 + z_2^2) + 2p_1 \left( 1 - \cos \frac{z_3}{2} \right)$$

(59)

Proceeding as before, we obtain

$$\dot{z}_3 = x_4 - \tilde{\alpha}_0$$

(60)

with

$$\tilde{\alpha}_0 = \frac{\partial x_0}{\partial z_2} (-z_1 + \varepsilon \sin x_3)$$

(61)

Therefore, (60) can be expressed in the form

$$\dot{z}_3 = z_4 + x_1 - \tilde{\alpha}_0$$

(62)

where we view $x_4$ as the virtual control and introduce $z_4 = x_4 - x_1$ as the last error variable.

Next, using (53), (54) and (62) we can write $\dot{V}_2$ as

$$\dot{V}_2 = p_0 (z_1 \dot{z}_1 + z_2 \dot{z}_2) + p_1 \sin \frac{z_3}{2} (z_4 + x_1 - \tilde{\alpha}_0)$$

$$= \varepsilon p_0 z_2 \sin x_0 + \varepsilon p_0 z_2 \left[ \sin (z_3 + x_0) - \sin x_0 \right] + p_1 \sin \frac{z_3}{2} (z_4 + x_1 - \tilde{\alpha}_0)$$

$$= p_1 \sin \frac{z_3}{2} \left[ z_4 + x_1 - \tilde{\alpha}_0 + \frac{2\varepsilon p_0}{p_1} z_2 \cos \left( x_0 + \frac{z_3}{2} \right) \right]$$

$$+ \varepsilon p_0 z_2 \sin x_0$$

(63)

To render $\dot{V}_2$ negative-semidefinite in $z_1$, $z_2$ and $z_3$ when $z_4 = 0$, we choose

$$x_1 = -\frac{2\varepsilon p_0}{p_1} z_2 \cos \left( x_0 + \frac{z_3}{2} \right) + \tilde{\alpha}_0 - c_1 \sin \frac{z_3}{2}$$

(64)

Then we include $z_4$ into our Lyapunov function

$$V = \frac{p_0}{2} (z_1^2 + z_2^2) + 2p_1 \left( 1 - \cos \frac{z_3}{2} \right) + \frac{p_2 z_4^2}{2}$$

(65)

which yields

$$\dot{V} = p_1 \sin \frac{z_3}{2} \left[ z_4 + x_1 - \tilde{\alpha}_0 + \frac{2\varepsilon p_0}{p_1} z_2 \cos \left( x_0 + \frac{z_3}{2} \right) \right]$$

$$+ p_2 z_4 \left( \dot{x}_4 - \tilde{\alpha}_1 \right) + \varepsilon p_0 z_2 \sin x_0$$

$$= \varepsilon p_0 z_2 \sin x_0 + p_1 \sin \frac{z_3}{2} \left[ x_1 - \tilde{\alpha}_0 + \frac{2\varepsilon p_0}{p_1} z_2 \cos \left( x_0 + \frac{z_3}{2} \right) \right] + p_2 z_4 \left( \frac{p_1}{p_2} \sin \frac{z_3}{2} + v - \tilde{\alpha}_1 \right)$$

(66)
where

\[
\bar{z}_1 = \begin{bmatrix}
\frac{\partial z_1}{\partial z_2} & \frac{\partial z_1}{\partial z_3}
\end{bmatrix}
\begin{bmatrix}
-z_1 + \varepsilon \sin x_3 \\
x_4 - \bar{z}_0
\end{bmatrix}
\]

(67)

Hence, the controller

\[
v = \bar{z}_1 - \frac{p_1}{p_2} \sin \frac{z_3}{2} - c_2 z_4
\]

(68)

yields

\[
\dot{V} = \varepsilon p_0 z_2 \sin x_0 - c_1 p_1 \sin^2 \frac{z_3}{2} - c_2 p_2 z_4^2
\]

(69)

which is negative-semidefinite. Using LaSalle’s invariance theorem, we can conclude that the closed-loop system has a globally asymptotically stable equilibrium at \(x_1 = x_2 = x_3 = x_4 = 0\). Therefore, this design not only eliminates the winding problem, as did the design of Section 3, but also further eliminates the existence of the second closed-loop stable equilibrium (the one with \(x_3 = \pi\)) and guarantees convergence of the actuator angle \(x_3\) to zero. To see this, note that combining (65) and (69) with LaSalle’s invariance theorem, we obtain that \(z_1, z_2, \sin z_3/2, z_4\) all converge to zero, which in turn implies that \(x_1 \to 0, x_2 \to 0, \sin x_3/2 \to 0, x_4 \to 0\). Hence, \(x_3\) converges to \(2n\pi\) with \(n\) being an integer. This shows that the winding problem is avoided, since \(x_3\) does not have to converge to zero even if it is viewed as taking values on \((-\infty, \infty)\). However, this is not the corrected interval for \(x_3\), because our true state-space manifold is \(\mathbb{R} \times \mathbb{R} \times [-\pi, \pi) \times \mathbb{R}\) which is cylindrical and hence not globally diffeomorphic to \(\mathbb{R}^4\). By redefining our state-space manifold accordingly, i.e. by viewing \(x_3\) as taking values on \([-\pi, \pi)\), we achieve two things:

1. The Lyapunov function \(V\) becomes globally positive-definite, monotonically increasing, and radially unbounded, provided that we choose \(c_0 \in (0, 2\pi)\). To see this, note that \(x_3 \in [-\pi, \pi)\) and \(z_0 \in [-c_0/2, c_0/2]\). Hence, if \(c_0 \in (0, 2\pi]\), \(z_3 = x_3 + x_0 \in [-2\pi, 2\pi]\) , and thus the term \((1 - \cos z_3/2)\) in the Lyapunov function becomes globally positive-definite and monotonically increasing.
2. The resulting closed-loop system has a globally asymptotically stable equilibrium at \(x_1 = x_2 = x_3 = x_4 = 0\).

5. CONCLUSIONS

The main point of this paper is that backstepping should not be viewed as a rigid design procedure, but rather as a design philosophy which can be bent and twisted to accommodate the specific needs of the system at hand. In the particular example of the benchmark nonlinear problem, we were able to exploit the flexibility of backstepping with respect to the selection of virtual controls, initial stabilizing functions and Lyapunov functions, to design three different controllers which use significantly smaller control effort than previous designs. Even in the ‘rigid’ design, the control effort was significantly decreased by proper gain selection. In addition, the other two designs eliminate the winding problem by using trigonometric functions in the selection of virtual controls or in the definition of the Lyapunov function.
A systematic method of exploiting all this flexibility would be very useful, but it is not yet available. Nevertheless, for any specific problem, design experience and understanding of the physical properties of the system can contribute a great deal towards this goal.

REFERENCES