



Robust SDRE filter design for nonlinear uncertain systems with an H_∞ performance criterion

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ABSTRACT

In order to remedy the effects of modeling uncertainty, measurement noise and input disturbance on the performance of the standard state-dependent Riccati equation (SDRE) filter, a new robust H_∞ SDRE filter design is developed in this paper. Based on the infinity-norm minimization criterion, the proposed filter effectively estimates the states of nonlinear uncertain system exposed to unknown disturbance inputs. Moreover, by assuming a mild Lipschitz condition on the chosen state-dependent coefficient form, fulfillment of a modified H_∞ performance index is guaranteed in the proposed filter. The effectiveness of the robust SDRE filter is demonstrated through numerical simulations where it brilliantly outperforms the conventional SDRE filter in presence of model uncertainties, disturbance and measurement noise, in terms of estimation error and region of convergence.

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1. Introduction

The state-dependent Riccati equation (SDRE) techniques are rapidly emerging as general design and synthesis methods of nonlinear feedback controllers and estimators for a broad class of nonlinear problems [1]. Essentially, the SDRE filter, developed over the past several years, is formulated by constructing the dual problem of the SDRE-based nonlinear regulator design technique [2]. The resulting observer has the same structure as the continuous steady state linear Kalman filter. In contrast to the EKF which uses the Jacobian of the nonlinearity in the system dynamics, the SDRE filter is based on parameterization that brings the nonlinear system to a linear-like structure with state-dependent coefficients (SDCs). As it is shown in [3], in the multivariable case, the SDC parameterization is not unique. Consequently, this method creates additional degrees of freedom that can be used to overcome the limitations such as low performance, singularities and loss of observability in a traditional estimation method [2]. Furthermore, such representation can fully capture the nonlinearities of the system, and therefore, this technique has been extensively used for nonlinear state/parameter estimation within aerospace [4,5] and power electronics applications [6,7].

There are two commonly used approaches for the SDRE filtering technique. The first approach, proposed originally by Mracek et al. in [2], is essentially constructed by considering the dual problem of the well-known SDRE nonlinear control law. The resulting filter has the same structure as the steady-state linear Kalman filter and the Kalman gain is obtained by solving a state dependent algebraic Riccati equation (SDARE) [2]. However, as reported in [8], this solution may be computationally expensive for large scale systems and depends significantly on the observability property of the system. The second approach that is recently suggested in the literature has the same structure as the linear Kalman filter [9,10]. Indeed, it removes the infinite time horizon assumption and requires the integration of a state-dependent differential Riccati equation (SDDRE) [9]. This alternative approach addresses the issues of high computational load and the restrictive observability requirement in the algebraic form of the estimator.

Although the effectiveness of the SDRE filter has been demonstrated through impressive simulation results, only few rigorous mathematical investigations on the filter have been considered in the literature [11,12]. Assuming certain observability and Lipschitz conditions on the SDC factorization and considering an incremental splitting of the state-dependent matrices, the local convergence of the continuous-time algebraic SDRE observer is proven in [11]. It is also shown in [1], how this observer asymptotically converges to the first-order minimum variance estimate given by the EKF. The analysis is based on stable manifold theory and Hamilton–Jacobi–Bellman (HJB) equations. Moreover, the

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analogous discrete-time difference observer is treated in [8,10], where two distinct sufficient conditions sets for its asymptotic stability are provided.

All the theoretical results cited above are confined to the nonlinear deterministic processes and they assume that the system model is *perfectly* known. Applying the standard SDRE filters to general stochastic systems, that is inevitable in practical purposes, requires accurate specification of the noise statistics as well. However, model uncertainty and incomplete statistical information are often encountered in real applications which may potentially give rise to excessive estimation errors. To tackle such difficulties, in this paper a robust SDRE filter is proposed with guaranteed H_∞ performance criterion. The motivation of this paper stems from the fact that in contrast to successful derivation of an H_∞ formulation concerning the SDRE control, accomplished by Cloutier et al. in [13], there is no documented similar attempt concerning its filtering counterpart.

Since the pioneer works of linear H_∞ filtering designs [14,15], the nonlinear H_∞ filtering problem has been studied by a number of authors (see [16,17] for a basic study, [18] for a general stochastic investigation, and [19,20] for approximate solutions). In this paper, a general continuous-time nonlinear uncertain model is considered as represented by Nguang and Fu, [16], to develop a robust H_∞ SDRE-based filter for nonlinear uncertain systems exposed to additive disturbance inputs. The proposed filter does not involve solving the Hamilton–Jacobi inequalities (HJIs) in [17,18], which is a time-consuming task. In addition, it obviates the need for linearization procedure of the extended H_∞ techniques [19], and the Riccati-based filtering design [20], and exhibits robustness against both unknown disturbances and model uncertainties. In fact in this paper the standard differential SDRE filter is reformulated into a robust filter such that the estimation error infinity norm is bounded and the filter achieves a prescribed level of disturbance attenuation for all admissible uncertainties. The key assumption made in this paper is that the SDC parameterization is chosen such that the state-dependent matrices are at least locally Lipschitz. Note that this result is substantially different from the well-established methods associated with the Lipschitz nonlinear systems, which decompose the entire model into a linear unforced part and a Lipschitz nonlinear uncertain part [21]. In comparison to the algorithms in [16–18], another advantage of the proposed method is its simplicity, as no complicated computation procedures are required to implement the estimator. It can be implemented systematically and inherits the elaborated capabilities of the SDRE-based filters [22], as well.

The rest of the paper is organized as follows. Section 2 provides the necessary backgrounds and formulations of the uncertain SDC description along with the robust SDRE filter design. In Section 3, by employing an appropriate Lyapunov function the performance index for the proposed filter is derived, which can be regarded as a modification of the conventional H_∞ performance criterion. Section 4 provides simulation examples to illustrate some definite superiority of the proposed filter over the corresponding usual SDRE-based filters. Finally, some conclusions are drawn in Section 5.

2. Robust SDRE filter and preliminaries

Consider a smooth nonlinear uncertain system described by continuous-time equations of the following form:

$$\dot{x}(t) = f(x) + \Delta f(x) + G(t)w_0(t) \quad (1)$$

$$y(t) = h(x) + \Delta h(x) + D(t)v_0(t) \quad (2)$$

where $x(t) \in R^n$ is the state, $y(t) \in R^m$ is the measured output, and $w_0(t) \in R^p$ and $v_0(t) \in R^q$ are process and measurement noises with unknown statistical properties, which stand for exogenous

disturbance inputs. For the sake of simplicity, we restrict ourselves to unforced noise-driven systems, a slightly more general representation than that of [16]. Some remarks on the forced case and affine in control input are given in Section 3. The nonlinear system dynamic $f(x)$ and the observation model $h(x)$ are assumed to be known as C^1 -functions. $G(t)$ and $D(t)$ are time varying known matrices of size $n \times p$ and $m \times q$, respectively. Also, $\Delta f(x)$ and $\Delta h(x)$ represent the system model uncertainties.

Assumption 1. Let the model uncertainties satisfy

$$\begin{bmatrix} \Delta f(x) \\ \Delta h(x) \end{bmatrix} = \begin{bmatrix} E_1(t)\Delta_1(t) \\ E_2(t)\Delta_2(t) \end{bmatrix} N(x) \quad (3)$$

in which $N(x) \in C^1$, $E_1(t)$ and $E_2(t)$ are known matrix functions with appropriate dimensions that characterize the structure of the uncertainties. Also, $\Delta_1(t)$ and $\Delta_2(t)$ are norm-bounded unknown matrices. By performing direct parameterization, the nonlinear dynamics (1) and (2) accompanied by Assumption 1 can be put into the following *uncertain* state-dependent coefficient (SDC) form

$$\dot{x}(t) = A(x)x + E_1(t)\Delta_1(t)N(x) + G(t)w_0(t) \quad (4)$$

$$y(t) = C(x)x + E_2(t)\Delta_2(t)N(x) + D(t)v_0(t) \quad (5)$$

where, $f(x) = A(x)x$, and $h(x) = C(x)x$. Note that the SDC parameterization is unique only if x is scalar [12] (also see the Remark 1 given below). Besides, the smoothness of the vector functions $f(x)$ and $h(x)$ with $f(0) = h(0) = 0$ makes it feasible [2,3] (see also [22] for effective handling of situations which prevent a straightforward parameterization).

Remark 1. If $A_1(x)$ and $A_2(x)$ are two distinct factorization of $f(x)$, then

$$A_3(x) = M(x)A_1(x) + (I - M(x))A_2(x)$$

is also a parameterization of $f(x)$ for each matrix-valued function $M(x) \in R^{n \times n}$. This is an exclusive characteristic of all SDRE-based design techniques, which has been successfully used not only to avoid singularity or loss of observability, but also to enhance performance (cf. [1,2,10]). Moreover, it may be used to satisfy the Lipschitz condition in our filtering design (see Remark 6).

Let us define the signal to be estimated as follows

$$z(x(t)) = L(t)x(t) \quad (6)$$

where $z(x) \in R^s$ can be viewed as the filter output, and $L(t)$ is a known $s \times n$ matrix bounded via

$$\underline{l} \leq L^T(t)L(t) \leq \bar{l} \quad (7)$$

for every $t \geq 0$ with some positive real numbers \underline{l}, \bar{l} .

We seek to propose a dynamic filter for the uncertain SDC model, given by (4) and (5), which robustly estimates the quantity $z(x)$ from the observed data $y(t)$ with a guaranteed H_∞ performance criterion. In other words, it is desired to ensure a bounded energy gain from the input noises ($w_0(t)$, $v_0(t)$) to the estimation error in terms of the H_∞ norm. The proposed filter has an SDRE-like structure and is prescribed to be

$$\dot{\hat{x}} = A(\hat{x})\hat{x} + K(t)[y(t) - C(\hat{x})\hat{x}] + \mu^{-2}P(t)\nabla_x N(\hat{x})^T N(\hat{x}). \quad (8)$$

In the above, $\mu > 0$ is a free design parameter and ∇_x denotes the gradient with respect to x . Furthermore, $\hat{x}(t)$ represents the estimated state vector, and the filter gain matrix, $K(t) \in R^{n \times m}$, is defined as

$$K(t) = P(t)C^T(\hat{x})R^{-1} \quad (9)$$

in the same way as for the usual SDRE filter. The positive definite matrix $P(t)$ is updated through the following state-dependent differential Riccati equation (SDDRE)

$$\begin{aligned} \dot{P}(t) = & A(\hat{x})P(t) + P(t)A^T(\hat{x}) + \Gamma(t)Q\Gamma^T(t) \\ & - P(t)[C^T(\hat{x})R^{-1}C(\hat{x}) - \mu^{-2}\nabla_x N(\hat{x})^T \nabla_x N(\hat{x}) \\ & - \lambda^{-2}\nabla_x z(\hat{x})^T \nabla_x z(\hat{x})]P(t) \end{aligned} \quad (10)$$

with

$$\Gamma(t) = [\mu E_1(t) \ G(t)], \quad (11)$$

positive definite matrix $Q \in R^{n \times n}$, symmetric positive definite matrix $R \in R^{m \times m}$, and a given positive real value $\lambda > 0$ that indirectly indicates the level of disturbance attenuation in our robust filter design. This variety as well as the exact role of the free parameter μ will be clarified in the next section.

Remark 2. It can be easily seen that with $\lambda, \mu \rightarrow \infty$, the proposed filter reverts to the standard differential SDRE filter [9]. Meanwhile, setting $\mu = \infty$ together with replacing $A(\hat{x})$ and $C(\hat{x})$ by the Jacobian of $f(x)$ and $h(x)$, respectively, in (9) and (10) render the structure of the extended H_∞ filter [19,23].

Before analyzing the performance of the robust SDRE filter, we recall two preparatory definitions within the H_∞ filtering theory.

Definition 1 (Extended L_2 -Space). The set $L_2[0, T]$ consists of all Lebesgue measurable functions $g(t) \in R^+ \rightarrow R^l$ such that

$$\int_0^T \|g(t)\|^2 dt < \infty \quad (12)$$

for every $T \geq 0$ with $\|g(t)\|$ as the Euclidean norm of the vector $g(t)$ (see, e.g., [17,21]).

Definition 2 (Robust H_∞ SDRE Filtering). Given any real scalar $\gamma > 0$, the dynamic SDRE filter (8)–(10) associated with the dynamics (4)–(6) is said to satisfy the H_∞ performance criterion if

$$\int_0^T \|z(t) - \hat{z}(t)\|^2 dt \leq \gamma^2 \int_0^T (\|w_0(t)\|_W^2 + \|v_0(t)\|_V^2) dt \quad (13)$$

holds for all $T \geq 0$, all $v_0(t), w_0(t) \in L_2[0, T]$, and all admissible uncertainties. Where, $\|w_0(t)\|_W$ and $\|v_0(t)\|_V$ are taken to be Euclidean norms scaled by some positive matrices W and V , respectively.

Remark 3. Inequality (13) implies that the L_2 -gain from the exogenous inputs ($w_0(t), v_0(t)$) to $z(t) - \hat{z}(t)$, called the generalized estimation error, is less than or equal to some minimum value γ^2 . It only necessitates that the disturbances have finite energy which is a familiar mild assumption.

Remark 4. Definition 2 is derived from what is declared by Nguang and Fu in [16]. The difference is that we consider two distinct noise sources with scaled Euclidean norms. These scalings, which are similar to those introduced in [19], may be interpreted as simple weights because H_∞ filtering does not rely on the availability of statistical information.

3. H_∞ performance analysis

In this section, we analyze the estimation error dynamics to derive an interesting feature of the proposed robust filter, which will be properly called modified H_∞ performance index. This criterion reveals the ability of the filter to minimize the effects of disturbances and uncertainties on the estimation error.

In order to facilitate our analysis, we adopt the following notation

$$\begin{aligned} w(t) &= [\mu^{-1}\Delta_1(t)N(x) \ w_0(t)]^T \\ v(t) &= [E_2(t)\Delta_2(t)N(x) \ v_0(t)]^T \end{aligned} \quad (14)$$

in which, the uncertainties are treated as bounded noise signals. The estimation error is defined by

$$e(t) = x(t) - \hat{x}(t). \quad (15)$$

Subtracting (8) from (4) and considering (13) and (11), the error dynamics is expressed as

$$\begin{aligned} \dot{e}(t) = & A(x)x + \Gamma(t)w(t) - A(\hat{x})\hat{x} \\ & - K(t)[y(t) - C(\hat{x})\hat{x}] - \mu^{-2}P(t)\nabla_x N(\hat{x})^T N(\hat{x}). \end{aligned} \quad (16)$$

Adding and subtracting $A(\hat{x})x$ to the whole equation, and adding and subtracting $C(\hat{x})x$ into the bracket lead to

$$\begin{aligned} \dot{e}(t) = & A(\hat{x})(x - \hat{x}) + (A(x) - A(\hat{x}))x \\ & - K(t)[C(\hat{x})(x - \hat{x}) + (C(x) - C(\hat{x}))x + v(t)] \\ & + \Gamma(t)w(t) - \mu^{-2}P(t)\nabla_x N(\hat{x})^T N(\hat{x}) \end{aligned} \quad (17)$$

rearranging the terms together with (15), we have

$$\begin{aligned} \dot{e}(t) = & [A(\hat{x}) - K(t)C(\hat{x})]e(t) + \alpha(x, \hat{x}) - K(t)\beta(x, \hat{x}) \\ & - K(t)v(t) + \Gamma(t)w(t) - \mu^{-2}P(t)\nabla_x N(\hat{x})^T N(\hat{x}) \end{aligned} \quad (18)$$

where the nonlinear functions $\alpha(x, \hat{x})$ and $\beta(x, \hat{x})$ are given by

$$\alpha(x, \hat{x}) = [A(x) - A(\hat{x})]x \quad (19)$$

$$\beta(x, \hat{x}) = [C(x) - C(\hat{x})]x. \quad (20)$$

The requirements for Theorem 1 given below, which embodies the main result of this paper, are summarized by the following assumptions.

Assumption 2. The state-dependent matrix $C(x)$ and the state vector $x(t)$ are bounded via

$$\|C(x)\| \leq \bar{c} \quad (21)$$

$$\|x(t)\| \leq \sigma \quad (22)$$

for all $t \geq 0$ and some positive real numbers $\bar{c}, \sigma > 0$.

Remark 5. Note that the assumption above is not severe. In particular, for many applications boundedness of the state variables, which often represent physical quantities, is natural. Besides, if $C(x)$ fulfill (21) for every physical reasonable value of the state vector $x(t)$, we may suppose without loss of generality that (21) holds.

Assumption 3. The SDC parameterization is chosen such that $A(x)$ and $C(x)$ are at least locally Lipschitz, i.e., there exist constants $k_A, k_C > 0$ such that

$$\|A(x) - A(\hat{x})\| \leq k_A \|x - \hat{x}\| \quad (23)$$

$$\|C(x) - C(\hat{x})\| \leq k_C \|x - \hat{x}\| \quad (24)$$

hold for all $x, \hat{x} \in R^n$ with $\|x - \hat{x}\| \leq \varepsilon_A$ and $\|x - \hat{x}\| \leq \varepsilon_C$, respectively.

It should be mentioned that if the SDC form fulfills the Lipschitz condition globally in R^n , then all the results in this section will be valid globally.

Remark 6. Inequalities (23) and (24) are the key conditions in our performance analysis. They are similar to Lipschitz conditions imposed in [11] and may be difficult to satisfy for some nonlinear dynamics. Nevertheless, additional degrees of freedom provided by nonuniqueness of the SDC parameterization can be exploited to realize Assumption 3.

With these prerequisites we are able to state the following theorem, which demonstrates how a modified H_∞ performance index is met by applying the proposed SDRE filter.

Theorem 1. Consider the nonlinear uncertain system of (4)–(6) along with the robust SDRE filter described by (8)–(10) with some $\lambda, \mu > 0$ and positive definite matrices Q and R . Under Assumptions 2 and 3 the generalized estimation error $z(t) - \hat{z}(t)$ fulfills a modified type of the H_∞ performance criterion introduced in Definition 2, provided that the SDDRE (10) has a positive definite solution for all $t \geq 0$ and λ is chosen such that

$$\lambda^{-2}l > 2\kappa \quad (25)$$

where,

$$\kappa = \left(\frac{k_A}{\underline{p}} + \frac{\bar{c}k_C}{\underline{r}} \right) \sigma \quad (26)$$

and $\underline{r}, \underline{p} > 0$ denote the smallest eigenvalue of the positive definite matrices R and $P(t)$, respectively. Furthermore, the disturbance attenuation level, γ in (13), is given by

$$\gamma^2 = \frac{\bar{l}}{\lambda^{-2}\underline{l} - 2\kappa} \quad (27)$$

with \bar{l}, \underline{l} in (7).

Remark 7. For usual differential SDRE filter, the solution of the standard SDDRE is positive definite and has an upper bound if the SDC form satisfies a certain uniform detectability condition as stated in [11]. Unfortunately, this condition cannot be applied to get similar results for the H_∞ -filtering-like SDDRE (10). However, it is a well-known problem arising in H_∞ control as well as H_∞ filtering that the solutions of the related Riccati equations suffered from lack of being positive definite (cf. [20]).

Remark 8. The existence of a positive definite solution $P(\cdot)$ for the SDDRE (10) depends mainly on an appropriate choice of λ and μ . To find suitable values for λ and μ one can employ a binary search algorithm, which is widely used to solve H_∞ control and H_∞ filtering problems (see, e.g., [20,21]).

Remark 9. Clearly, the filter attenuation constant γ is indirectly specified by the design parameter λ while it is independent of μ . The extra design parameter μ has turned out to be very useful for ensuring solvability of (10) with the desired positive definiteness property. Furthermore, it scales the uncertainty norm in the proposed performance index (see the proof of Theorem 1).

Remark 10. Inequality (25) roughly means that λ must be chosen sufficiently small. This is in accordance with the purpose of performance improvement which calls for a small value of γ in (27). Furthermore, it can be shown that (25) is obviated while the estimation error still assures the same performance index with different attenuation constant $\gamma^2 = \lambda^2(2\bar{l}/\underline{l})$, if inequalities (23) and (24) are replaced by more restricted Lipschitz conditions of order two, e.g., $\|A(x) - A(\hat{x})\| \leq k_A \|x - \hat{x}\|^2$. The proof of theorem can be modified easily for this case.

To prove Theorem 1 the following lemma is required.

Lemma 1. Let inequalities (21)–(24) be satisfied, then for an estimation error $\|e\| \leq \varepsilon$, $\Pi(t) = P(t)^{-1}$ satisfies the following inequality

$$(x - \hat{x})^T \Pi(t) [\alpha(x, \hat{x}) - K(t)\beta(x, \hat{x})] \leq \kappa \|x - \hat{x}\|^2 \quad (28)$$

where $\varepsilon = \min(\varepsilon_A, \varepsilon_C)$. The positive real scalar κ , the matrix $K(t)$, and the nonlinearities α, β are given by (26), (9), (19), and (20) respectively.

Proof. Applying the triangle inequality, $K = PC^T(\hat{x})R^{-1}$ and $\Pi P = I$ leads to

$$\begin{aligned} & \|(x - \hat{x})^T \Pi \alpha(x, \hat{x}) - (x - \hat{x})^T \Pi K \beta(x, \hat{x})\| \\ & \leq \|(x - \hat{x})^T \Pi \alpha(x, \hat{x})\| + \|(x - \hat{x})^T C^T(\hat{x})R^{-1} \chi(x, \hat{x})\|. \end{aligned} \quad (29)$$

In view of the Lipschitz conditions on the SDC form and inequality (22), the nonlinear functions α, β are bounded via

$$\|\alpha(x, \hat{x})\| = \| [A(x) - A(\hat{x})]x \| \leq k_A \sigma \|x - \hat{x}\| \quad (30)$$

$$\|\beta(x, \hat{x})\| = \| [C(x) - C(\hat{x})]x \| \leq k_C \sigma \|x - \hat{x}\| \quad (31)$$

with $\|x - \hat{x}\| \leq \varepsilon_A$ and $\|x - \hat{x}\| \leq \varepsilon_C$, respectively. Choosing $\varepsilon = \min(\varepsilon_A, \varepsilon_C)$ and employing (30), (31), (21), $\|\Pi\| \leq 1/\underline{p}$, and $\|R^{-1}\| \leq 1/\underline{r}$ in (29), we obtain

$$\begin{aligned} & \|(x - \hat{x})^T \Pi \phi(x, \hat{x}, u) - (x - \hat{x})^T \Pi K \chi(x, \hat{x})\| \\ & \leq \|x - \hat{x}\| \frac{k_A \sigma}{\underline{p}} \|x - \hat{x}\| + \|x - \hat{x}\| \frac{\bar{c}k_C \sigma}{\underline{r}} \|x - \hat{x}\| \end{aligned} \quad (32)$$

therefore, (28) follows immediately with κ given in (26). \square

Proof of Theorem 1. Choose a Lyapunov function as

$$V(e(t)) = e^T(t) \Pi(t) e(t) \quad (33)$$

with $\Pi(t) = P(t)^{-1}$, which definitely is positive definite since $P(t)$ in (10) is positive definite. Take time derivative of $V(e)$

$$\dot{V}(e(t)) = \dot{e}^T(t) \Pi(t) e(t) + e^T(t) \dot{\Pi}(t) e(t) + e^T(t) \Pi(t) \dot{e}(t). \quad (34)$$

Insert (18) and (10) in (34) along with considering $\dot{\Pi}(t) = -\Pi(t) \dot{P}(t) \Pi(t)$, yield with a few rearrangement to

$$\begin{aligned} \dot{V}(e(t)) &= e^T [-\lambda^{-2} L^T L] e + 2e^T \Pi [\alpha - K\beta] \\ &+ w^T \Gamma^T \Pi e + e^T \Pi \Gamma w - v^T R^{-1} C(\hat{x}) e \\ &- e^T C^T(\hat{x}) R^{-1} v - e^T C^T(\hat{x}) R^{-1} C(\hat{x}) e \\ &+ \mu^{-2} \{ -e^T \nabla_x N(\hat{x})^T N(\hat{x}) \\ &- e^T \nabla_x N(\hat{x})^T N(\hat{x}) - (\nabla_x N(\hat{x})^T N(\hat{x}))^T e \}. \end{aligned} \quad (35)$$

Let us set $s = Q^{-1/2} w - (HQ^{1/2})^T \Pi e$ and $\eta = v + C(\hat{x})e$, then (35) can be rewritten as

$$\begin{aligned} \dot{V}(e(t)) &= e^T [-\lambda^{-2} L^T L] e + 2e^T \Pi [\alpha - K\beta] \\ &+ w^T Q^{-1} w - s^T s + v^T R^{-1} v - \eta^T R^{-1} \eta \\ &+ \mu^{-2} \{ -e^T [\nabla_x N(\hat{x})^T \nabla_x N(\hat{x})] e \\ &- e^T \nabla_x N(\hat{x})^T N(\hat{x}) - (\nabla_x N(\hat{x})^T N(\hat{x}))^T e \} \end{aligned} \quad (36)$$

where $Q = Q^{(1/2)} Q^{(1/2)T}$, $R = R^{(1/2)} R^{(1/2)T}$. Completing the square in the accolade of (36) and use triangular inequality, to obtain by virtue of Lemma 1.

$$\begin{aligned} \dot{V}(e(t)) &\leq e^T [-\lambda^{-2} L^T L] e + 2\kappa \|e\|^2 \\ &+ w^T Q^{-1} w + v^T R^{-1} v + \mu^{-2} N^T(\hat{x}) N(\hat{x}). \end{aligned} \quad (37)$$

This holds, provided that the estimation errors satisfy $\|e\| \leq \varepsilon$, in which, $\varepsilon = \min(\varepsilon_A, \varepsilon_C)$.

The use of $-\|l\| \leq -L^T L \leq \|l\|$ leads to

$$\begin{aligned} \dot{V}(e(t)) &\leq -\frac{\lambda^{-2}l - 2\kappa}{\bar{l}} e^T [L^T L] e \\ &+ w^T Q^{-1} w + v^T R^{-1} v + \mu^{-2} N^T(\hat{x}) N(\hat{x}). \end{aligned} \quad (38)$$

By integrating both sides of (38) over the time interval $[0, T]$, the H_∞ performance index of the proposed filter is derived as

$$\begin{aligned} \int_0^T \|L(t)e(t)\|^2 dt &\leq \gamma^2 \int_0^T (\|w(t)\|_{Q^{-1}}^2 + \|v(t)\|_{R^{-1}}^2 \\ &+ \|N(\hat{x})\|_{\mu^{-2}}^2 + e^T(0) \Pi(0) e(0)) dt \end{aligned} \quad (39)$$

where $\gamma^2 = \bar{l}/(\lambda^{-2l} - 2\kappa)$ is a positive real number if $\lambda^{-2l} > 2\kappa$, and indicates the filter attenuation constant. Clearly, (39) can be viewed as a modification of (13) in the sense that it incorporates the effects of model uncertainties and initial estimation errors. This concludes the proof of theorem. \square

Remark 11. Note that, γ is not only an index to indicate the disturbance attenuation level, but also it is an important parameter describing filter estimation ability in the worst case. In other words, decreasing γ will enhance the robustness of the filter. The SDRE-based H_∞ control offered in [13], is based on a game theoretic approach and exhibits robustness only against disturbances. However, the significance of (39) is that it is derived from Lyapunov-based approach and guarantees robustness against the system model uncertainty as well as the process and measurement noises.

We now endeavor to extend our results to a class of uncertain forced systems. Suppose the state equation (4) is controlled by the input $u(t) \in R^l$ as follows

$$\dot{x}(t) = A(x)x + B(x)u + E_1(t)\Delta_1(t)N(x) + G(t)w_0(t) \quad (40)$$

where $B(x) \in R^{n \times l}$ is a known matrix function. Eq. (40) together with (5) represents an uncertain form of the nonlinear affine system used in the SDRE control technique. We claim that, under certain conditions, the proposed filter will successfully work for the given forced system, as well. The following corollary evolves this fact.

Corollary 1. Let the control input $u(t)$ is norm-bounded, i.e., $\|u(t)\| \leq \rho$. Furthermore consider some $\rho > 0$, and a locally Lipschitz control matrix $B(x)$, for $k_B > 0$ and $\|x - \hat{x}\| \leq \varepsilon_B$, i.e. $\|B(x) - B(\hat{x})\| \leq k_B\|x - \hat{x}\|$. Then under the conditions of Theorem 1, applying (8)–(10), with an additive term of $B(\hat{x})u$ in (8), to the given system (40) and (5) achieves the same performance index as (40). The only difference to previous result is that in this case, $\kappa = (k_A\sigma + k_B\rho)/p + \bar{c}k_C\sigma/r$ and $\varepsilon = \min(\varepsilon_A, \varepsilon_B, \varepsilon_C)$.

Proof. The proof is in complete analogy to that of Theorem 1. \square

4. Illustrative examples

4.1. Example 1

To illustrate the performance improvement of the proposed SDRE filter over the usual algebraic and differential SDRE filters, we consider a second-order nonlinear uncertain system expressed as

$$\dot{x}(t) = \begin{bmatrix} x_1^2 - 2x_1x_2 + (-1 + \delta_1(t))x_2 \\ x_1x_2 + x_2 \sin x_2 + (1 + \delta_2(t))x_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_0(t) \quad (41)$$

$$y(t) = x_1 + v_0(t) \quad (42)$$

where, $x = [x_1 \ x_2]^T$ and $\delta_1(t)$, $\delta_2(t)$ are unknown time varying functions satisfying the following condition

$$\left\| \begin{bmatrix} \delta_1(t) \\ \delta_2(t) \end{bmatrix} \right\| \leq 1. \quad (43)$$

The disturbing noise signals $w_0(t)$ and $v_0(t)$ are determined from two different distributions with unknown statistics, in which $w_0(t)$ is a white Gaussian process noise, while $v_0(t)$ is a uniformly distributed measurement noise.

Obviously, (41) and (42) have the general form of (1) and (2) and can be rewritten to the uncertain SDC form (4) and (5) by any

suitable parameterization. Among several possible choices, let us set

$$A(x) = \begin{bmatrix} x_1 - 2x_2 & -1 \\ -1 & x_1 + \sin x_2 \end{bmatrix} \quad (44)$$

$$C(x) = [1 \ 0]. \quad (45)$$

In addition, in this example there is no measurement uncertainty ($\Delta_2 \equiv 0$) while the state equation uncertainty is described by

$$\Delta_1(t) = \begin{bmatrix} 0 & \delta_1(t) \\ \delta_2(t) & 0 \end{bmatrix}, \quad N(x) = x \quad (46)$$

with $E_1(t) = I_2$.

Note that (45) is a trivial choice, and one can choose other forms such as $C(x) = [1 + x_2 \ x_1]$. The reason of our choices, (44) and (45), is mainly related to the compliance of inequalities (21), (23), and (24). Firstly, it follows from (44) that for all $x, \hat{x} \in R^2$,

$$A(x) - A(\hat{x}) = \begin{bmatrix} (x_1 - \hat{x}_1) - 2(x_2 - \hat{x}_2) & 0 \\ 0 & (x_1 - \hat{x}_1) + (\sin x_2 - \sin \hat{x}_2) \end{bmatrix}. \quad (47)$$

Since

$$|(x_1 - \hat{x}_1) - 2(x_2 - \hat{x}_2)| \leq \sqrt{5}\|x - \hat{x}\|$$

and

$$|(x_1 - \hat{x}_1) + (\sin x_2 - \sin \hat{x}_2)| \leq \sqrt{2}\|x - \hat{x}\|,$$

it can be deduced that (23) is globally satisfied with $k_A = \sqrt{5}$. Secondly, the selected output matrix (45) fulfills (21) with $\bar{c} = 1$ and (24) with any positive real Lipschitz constant such as $k_C = 0.001$. Considering these facts in addition to the Lyapunov stability of (41), we may conclude that Assumptions 2 and 3 hold. Next, implement the robust SDRE filter according to (8)–(10) to obtain the desired performance for the system (41) and (42). The differential equations are solved numerically by the Runge–Kutta method, choosing the initial conditions $x(0) = [-0.5 \ 0.5]^T$ for the system to be observed, $\hat{x}(0) = [0.5 \ -0.5]^T$ for the filter and $P(0) = 10I_2$ for the SDDRE (10). The design details are summarized as follows.

The filter output, $z(x)$ in (6), is assumed to be $x(t)$ itself. Therefore, in this case, $L(t)$ is the identity matrix and \bar{l}, \bar{l} may be considered as one. Also choose the weighting matrices $Q = I_2$ and $R = 0.1$. The appropriate values for λ and μ are obtained using a binary search algorithm similar to that proposed in [12]. By this means, it turns out that $\lambda = 0.5$ and $\mu = 0.004$ are sufficient for $P(t)$ in (10) to be always positive definite. Besides, this value of λ will satisfy (25). This fact can be easily verified by inserting the values of k_A , k_C , and \bar{c} , which are analytically determined above, along with $r = 0.1$, $\sigma = 0.707$, and $p = 10$ into (25) which yields to $\kappa = 0.165$.

So far, all the sufficient conditions in Theorem 1 have been ensured, and hence, it is expected to reach to the modified H_∞ performance index obtained in (39). This is verified through the simulation results depicted in Fig. 1. The figure shows the true state of the system together with the estimated value obtained from the robust SDRE filter. It is clear that the filter performs as expected, and the estimated signals converge quickly to the corresponding actual ones in spite of the considered disturbances and modeling uncertainties. Note that according to (27), the given $\lambda = 0.5$ guarantees an attenuation level of $\gamma^2 = 0.27$. This means the energy gain from the disturbances to the estimation errors is bounded by 0.27.

For the sake of comparison, the standard algebraic and differential SDRE filters, namely SDARE filter and SDDRE filter, were also simulated with the same weighting matrices Q, R and the same initial conditions $\hat{x}(0)$. The results of this simulation are illustrated in Fig. 2. It is observed that in these two cases, the estimated signals do not track the true ones and exhibit divergence.

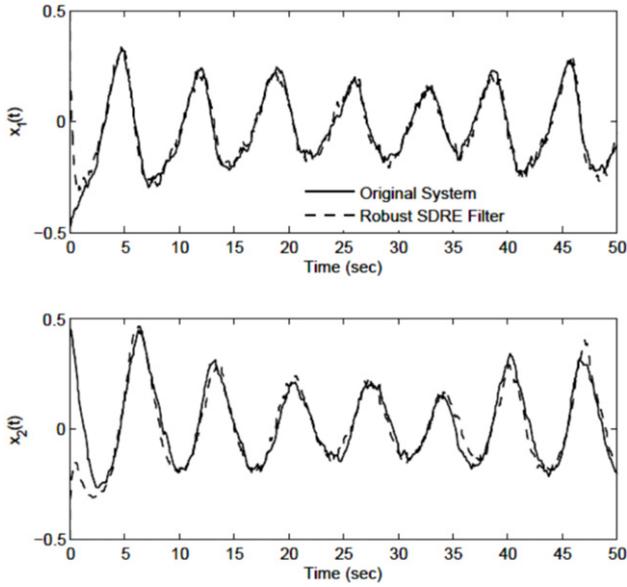


Fig. 1. The actual and the estimated states by the robust SDRE filter.

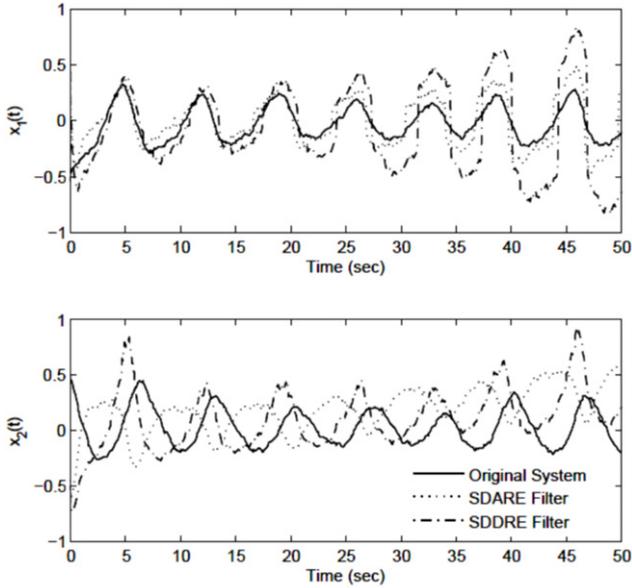


Fig. 2. The actual and the estimated states by the algebraic and differential SDRE filters.

4.2. Example 2 (induction motor).

In order to show the effectiveness of the proposed filtering scheme, let us implement it for the estimation of flux and angular velocity of an induction motor. The normalized state equation of such system may be written as follows [24,25]. Note that in this example a forced system is considered for the robust SDRE filter implementation.

$$\begin{aligned}
 \dot{x}_1(t) &= k_1x_1(t) + u_1(t)x_2(t) + k_2x_3(t) + u_2(t) \\
 \dot{x}_2(t) &= -u_1(t)x_1(t) + k_1x_2(t) + k_2x_4(t) \\
 \dot{x}_3(t) &= k_3x_1(t) + k_4x_3(t) + (u_1(t) - x_5(t))x_4(t) \\
 \dot{x}_4(t) &= k_3x_2(t) - (u_1(t) - x_5(t))x_3(t) + k_4x_4(t) \\
 \dot{x}_5(t) &= k_5(x_1(t)x_4(t) - x_2(t)x_3(t)) + k_6u_3(t).
 \end{aligned} \tag{48}$$

In the above, x_1, x_2 and x_3, x_4 are the components of the stator and the rotor flux, respectively, and x_5 is the angular velocity. The inputs are denoted by u_1 as the frequency u_2 as the amplitude of the

stator voltage, and u_3 as the load torque. Furthermore, k_1, \dots, k_6 are some constants determined from the structure of the motor and its drive system [24]. The output equations are given as

$$\begin{aligned}
 y_1(t) &= k_7x_1(t) + k_8x_3(t) \\
 y_2(t) &= k_7x_2(t) + k_8x_4(t)
 \end{aligned} \tag{49}$$

in which, k_7 and $8 k_8$ are user defined parameters, to generate the normalized stator current denoted by $y_1(t)$ and $y_2(t)$. Consider the following SDC parameterization of (48) and (49)

$$A(x) = \begin{bmatrix} k_1 & 0 & k_2 & 0 & 0 \\ 0 & k_1 & 0 & k_2 & 0 \\ k_3 & 0 & k_4 & -x_5 & 0 \\ 0 & k_3 & 0 & k_4 & x_3 \\ k_5x_4 & -k_5x_3 & 0 & 0 & 0 \end{bmatrix} \tag{50}$$

$$B(x) = \begin{bmatrix} x_2 & 1 & 0 \\ -x_1 & 0 & 0 \\ x_4 & 0 & 0 \\ -x_3 & 0 & 0 \\ 0 & 0 & k_6 \end{bmatrix} \tag{51}$$

$$C(x) = \begin{bmatrix} k_7 & 0 & k_8 & 0 & 0 \\ 0 & k_7 & 0 & k_8 & 0 \end{bmatrix}. \tag{52}$$

Inequality (24) is evidently satisfied by the above output matrix (52). But, confirming the Lipschitz condition for the matrices (50) and (51) necessitate some calculations. For matrix $B(x)$ we have

$$\begin{aligned}
 \|B(x) - B(\hat{x})\| &= \sqrt{(x_1 - \hat{x}_1)^2 + (x_2 - \hat{x}_2)^2 + (x_3 - \hat{x}_3)^2 + (x_4 - \hat{x}_4)^2} \\
 &\leq k_B \|x - \hat{x}\|
 \end{aligned} \tag{53}$$

with $x, \hat{x} \in \mathbb{R}^5$. Therefore, Lipschitz condition for the matrices (51) is met for any positive real number k_B . Likewise, for the system matrix $A(x)$ we may derive:

$$\|A(x) - A(\hat{x})\| = \max(|x_3 - \hat{x}_3|, |x_5 - \hat{x}_5|, k_5\sqrt{(x_3 - \hat{x}_3)^2 + (x_4 - \hat{x}_4)^2}). \tag{54}$$

Hence, it can be concluded that Lipschitz condition for the matrices (50) is also satisfied for $k_A = \max(1, k_5)$.

We now proceed to implement the proposed robust SDRE filter. The following parameters are set in the simulations. $k_1 = -0.186, k_2 = 0.176, k_3 = 0.225, k_4 = -0.234, k_5 = -0.1081, k_6 = -0.018, k_7 = 4.643, k_8 = -4.448$. The control input is considered as $u(t) = [1 \ 1 \ 0]^T$, and the initial state conditions are set to: $x(0) = [0.2 \ -0.6 \ -0.4 \ 0.1 \ 0.3]^T$.

Let us consider a relatively large initial estimation error by choosing $\hat{x}(0) = [0.5 \ 0.1 \ 0.3 \ -0.2 \ 4]^T$, and design the proposed robust SDRE filter with the following design parameters: $Q = 0.04I_5, R = 0.06I_2, \lambda^2 = 0.7, \mu^2 = 0.008$.

For sake of comparison, the standard SDDRE filter, is also simulated with the same weighting matrices Q, R and the same initial conditions $\hat{x}(0)$. The results of these simulations are illustrated in Fig. 3. As it is observed in this figure the estimation error for the angular velocity estimates for the robust SDRE filter converges to zero, while the estimation error diverges in the case of SDDRE filter. This indicates that a larger region of convergence is attainable for robust SDRE filter in this case.

Furthermore, in order to illustrate that the conditions given in Theorem 1, are just sufficient conditions, the Lyapunov function (33) is evaluated and plotted in Fig. 4. It can be seen that although the Lyapunov function is not monotonically decreasing, the estimation error converges to zero. This reveals the fact that the obtained results are not necessary conditions for error decay.

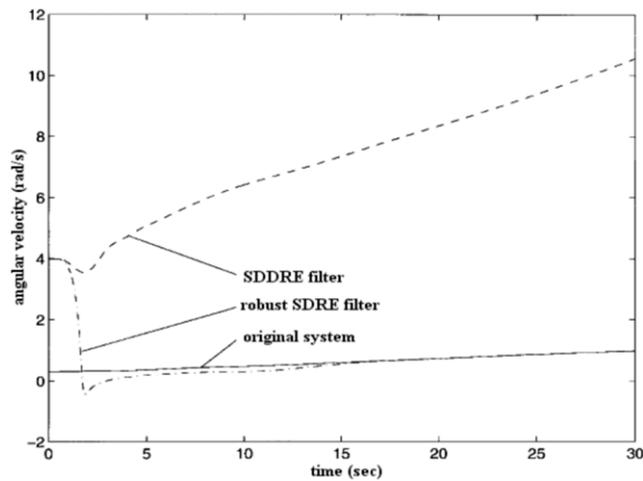


Fig. 3. The actual angular speed and its estimates by SDDRE filter and the proposed robust SDRE filter for large initial error.

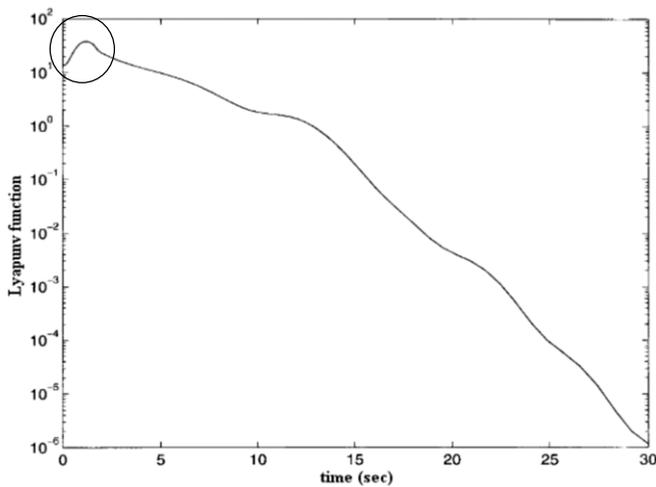


Fig. 4. The values of the Lyapunov function (33) of the proposed robust SDRE filter for large initial error.

5. Conclusion

To overcome the destructive effects of uncertain dynamics and unknown disturbance inputs on the performance of the usual SDRE filters a new robust H_∞ filter design is developed in this paper. The proposed filter can be systematically applied to nonlinear continuous-time systems with an uncertain SDC form. It is proved that under specific conditions the proposed filter guarantees the modified H_∞ performance criterion by choosing an appropriate Lyapunov function. This criterion is modified in the sense that it incorporates both the effects of disturbances and model uncertainties in the H_∞ norm minimization. Numerical simulations show the promising performance of the robust SDRE filter in comparison to the standard SDRE filters, in terms of estimation error and region of convergence. The obtained results

nominate the proposed filter as a viable H_∞ filtering method for practical applications.

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